eman ta zabal zazu

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Profinite R-analytic groups

Talde R-analitiko profinituak

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Abstract

While the theory of analytic groups is extensively developed over the p-adic numbers, relatively little is known about groups that are analytic over alternative coefficient rings such as $\mathbb{Z}_p[[t_1, \ldots, t_m]]$ or $\mathbb{F}_p[[t_1, \ldots, t_m]]$. This thesis is a contribution to the theory of analytic groups over general pro-p domains by means of a systematic investigation of structural concepts of that theory, including associated Lie algebras, submanifolds and analytic quotients.

Moreover, we study several group-theoretical properties in the setting of analytic groups, namely linearity, word problems and fractal dimensions. With regard to the first two properties –linear representations and word-conciseness of some analytic groups– we extend results that are well-established in the p-adic case. In contrast, the study of the Hausdorff dimension –where we mainly focus on groups that are analytic over $\mathbb{F}_p[[t]]$– shows significant differences between groups that are analytic over the p-adic numbers and those that are analytic over other coefficient rings.

Laburpena

Talde analitikoen teoria aski garatuta dago zenbaki p-adikoen gainean, baina alderatuta ezer gutxi ezagutzen da beste koefiziente eraztun batzuen, $\mathbb{Z}_p[[t_1, \ldots, t_m]]$ edo $\mathbb{F}_p[[t_1, \ldots, t_m]]$ kasu, gainean analitikoak diren taldeei buruz. Tesi hau pro-p domeinu orokorren gainean analitikoak diren taldeen inguruko hainbat ekarpenek osatzen dute. Alde batetik, teoria horretako egiturazko kontzeptu anitz xeheki aztertzen dira, esate baterako, elkarturiko Lieren aljebrak, azpibarietateak eta zatidura analitikoak.

Beste alde batetik, talde analitikoen testuinguruan talde teoriako zenbait propietate aztertzen dira, linealtasuna, hitz-problemak eta dimentsio fraktalak hain zuzen ere. Lehenengo bi gaiei dagokienez –talde analitiko batzuen adierazpen linealak eta hitz-laburtasuna–, erdietsiriko emaitzek kasu p-adikoan ezagunak diren teoremak orokortzen dituzte. Alabaina, Hausdorffen dimentsioaren azterketan –eskuarki $\mathbb{F}_p[[t]]$-ren gainean analitikoak diren taldeetara mugatuko gara– alde nabarmena dago zenbaki p-adikoen gainean eta beste koefiziente eraztun batzuen gainean analitikoak diren taldeen artean.

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Part I

Profinite R-analytic groups

Notazio indizea

Conventions. We assume that all rings are commutative and with identity. Moreover, throughout all the thesis p stands for a prime number, and q for a power of p.

Notation. Most of the notation is standard, except for $A^{(n)}$, which stands for the nth Cartensian power of the set A (we shall use this notation, since it will be common to write expressions of the form $(\mathfrak{a}^n)^{(m)}$ for an ideal $\mathfrak{a}$, so it is convenient to distinguish the nth Cartesian power $\mathfrak{a}^{(n)}$ from the nth power ideal $\mathfrak{a}^n$). Moreover, if $f\colon A \to B$ is a map, we denote by $f^{(n)}$ the map $A^{(n)} \to B^{(n)}$, $(a_1, \ldots, a_n) \mapsto (f(a_1), \ldots, f(a_n))$.

The remaining terminology is listed below:

$\mathbb{N}$	the natural numbers
$\mathbb{N}_0$	the natural numbers together with 0
$\mathbb{Z}$	the integers
$\mathbb{Z}_p$	the p-adic integers
$\mathbb{Q}$	the rational numbers
$\mathbb{Q}_p$	the p-adic numbers
$\mathbb{R}$	the real numbers
$\mathbb{R}_{\geq 0}$	the non-negative real numbers
$\mathbb{C}$	the complex numbers
$\mathbb{F}_q$	the finite field of size q
$\log_a$	the logarithm of basis a

$\mathcal{P}(A)$	the parts of A
$A \times B$	the Cartesian product of A and B
$\prod_{i \in I} A_i$	the Cartesian product of the directed family $\{A_i\}_{i \in I}$
$A^{(k)}$	the kth Cartesian power of A
$A \oplus B$	the direct sum of A and B
$A \ltimes B$	the semidirect product of A and B
$\bigoplus_{i \in I} A_i$	the direct sum of the directed family $\{A_i\}_{i \in I}$
$\prod_{i \in I} A_i / \mathcal{U}$	the ultraproduct of the directed family $\{A_i\}_{i \in I}$
$A^{\mathcal{U}}$	the ultrapower of A

$A \subseteq B$	A is a subset of B
$A \subseteq_o B$	A is an open subset of B
$A \subseteq_c B$	A is a closed subset of B
$A \leq B$	A is a subgroup of B
$A \leq_o B$	A is an open subgroup of B
$A \leq_c B$	A is a closed subgroup of B
$A \trianglelefteq B$	A is a normal subgroup of B
$A \trianglelefteq_o B$	A is a normal open subgroup of B
$A \trianglelefteq_c B$	A is a normal closed subgroup of B
$A \operatorname{char} B$	A is a characteristic subgroup of B

Let G be a group and let $g, x, y \in G$:

x^y	$y^{-1}xy$
$[x, y]$	$x^{-1}y^{-1}xy$
$[x_1, \dots, x_n]$	$[[x_1, \dots, x_{n-1}], x_n]$
$Z(G)$	the centre of G
$C_G(g)$	the centraliser of $g \in G$
$G' = [G, G]$	the derived subgroup of G : $\langle [x, y] \mid x, y \in G \rangle$.
G^n	the nth power subgroup: $\langle g^n \mid g \in G \rangle$
$[H_1, \dots, H_n]$	$\langle [h_1, \dots, h_n] \mid h_i \in H_i \rangle$
c_y	the conjugation isomorphism $G \to G$, $x \mapsto x^y$
L_y	the left multiplication map $G \to G$, $x \mapsto yx$
R_y	the right multiplication map $G \to G$, $x \mapsto xy$

$\ker f$	the kernel of the group (resp. ring) homomorphism f
$\operatorname{im} f$	the image of the group (resp. ring) homomorphism f
$\operatorname{Hom}(A, B)$	group (resp. ring) homomorphisms $f\colon A \to B$.

Let Q be a ring:

$\mathcal{U}(Q)$	the units of Q
$\operatorname{char} Q$	the characteristic of Q
$\dim_{\mathrm{Krull}} Q$	the Krull dimension of Q
$\operatorname{Frac}(Q)$	the fraction field of the integral domain Q
$Q[[t_1, \dots, t_m]]$	the ring of formal power series in m variables and coefficients in Q

$\mathrm{M}_{n\times m}(Q)$	$n \times m$ matrices with coefficients in Q
$\mathrm{M}_n(Q)$	$n \times n$ matrices with coefficients in Q
$\mathrm{GL}_n(Q)$	general linear group with coefficients in Q
$\mathrm{SL}_n(Q)$	special linear group with coefficients in Q
$\mathrm{SO}_n(Q)$	special orthogonal group with coefficients in Q
$\mathrm{Sp}_n(Q)$	symplectic group with coefficients in Q
$\mathcal{U}_n(Q)$	upper triangular matrices with coefficients in Q

Let K be a field:

K^{alg}	the algebraic closure of K
$\dim_K$	K-vector space dimension

Let M be an R-analytic manifold:

$\dim_x M$	the analytic dimension of the manifold M at x
$\dim G$	analytic dimension of the analytic group G

Let $\mathcal{G}$ be a linear algebraic group:

$R_u(\mathcal{G})$	unipotent radical

rk	rank of a matrix
res. rk	residual rank of a matrix
det	determinant of a matrix
tr	trace of a matrix

$D\mathbf{U}$	differential of a tuple of power series
$\mathcal{J}_x F$	the Jacobian matrix of F at x

$\mathrm{rk}\, M$	the rank of a free module M
$\mathrm{Iso}_M(N)$	isolator of N in M
$\mathrm{End}_R(M)$	endomorphisms of the R-module M

hdim	Hausdorff dimension
hspec	Hausdorff spectrum
$\mathrm{hdim}_{\mathrm{st}}$	standard Hausdorff dimension
$\mathrm{hspec}_{\mathrm{st}}$	standard Hausdorff spectrum
bdim	box dimension
$\mathrm{bdim}_{\mathrm{st}}$	standard box dimension
lbdim	lower box dimension
$\mathrm{lbdim}_{\mathrm{st}}$	standard lower box dimension
ubdim	upper box dimension
$\mathrm{ubdim}_{\mathrm{st}}$	standard upper box dimension

$w\{G\}$	the set of w-values of G
$w(G)$	the verbal subgroup of w
$w^*(G)$	the marginal subgroup of w
$X^{*\ell}$	the set of products of ℓ elements of $X \cup X^{-1} \cup \{1\}$

$\deg \mathfrak{L}$	degree of the Lie algebra $\mathfrak{L}$
$\deg \phi$	degree of the representation ϕ
$Z(\mathfrak{L})$	centre of the Lie algebra $\mathfrak{L}$
$R_n(\mathfrak{L})$	nilpotent radical of $\mathfrak{L}$
$R_s(\mathfrak{L})$	soluble radical of $\mathfrak{L}$
$\mathbf{T}_R(\mathfrak{L})$	torsion algebra of $\mathfrak{L}$
$\mathcal{U}_R(\mathfrak{L})$	universal enveloping algebra of $\mathfrak{L}$
$\mathrm{Der}_R(\mathfrak{L})$	derivations of $\mathfrak{L}$
$\mathrm{Cent}(\mathfrak{L})$	centroid of $\mathfrak{L}$

"I tend to think too much.
My greatest successes came from decisions I made when I stopped thinking and simply did what felt right.
Even if there was no good explanation for what I did. [...]
Even if there were very good reasons for me not to do what I did."

Kvothe, (Patrick Rothfuss, *The Name of the Wind*)

Introduction

This dissertation is a monograph on analytic groups. These comprise an abstract group together with an analytic manifold structure over a convenient topological ring in such a way that both structures are compatible, in the sense that the multiplication map and the inversion map are analytic functions.

The theory of analytic Lie groups over topological fields is a source of examples of profinite groups. Of course, over the classical fields $\mathbb{R}$ and $\mathbb{C}$, analytic Lie groups cannot be profinite unless they are finite, as they should be both compact and locally homeomorphic to a totally disconnected subset of $\mathbb{C}^{(n)}$. However, profinite analytic groups might arise if the underlying group of the base ring is a profinite group in its own right. For instance, in the treatise *Groupes analytiques p-adiques* [50], Lazard extensively studied the p-adic analytic groups, that is, analytic Lie groups over the field of p-adic numbers $\mathbb{Q}_p$ – equivalently, over the valuation ring of p-adic integers $\mathbb{Z}_p$–; and he showed that compact p-adic analytic groups are actually profinite groups.

In addition to the original purely analytic point of view, there are several alternative characterisations of p-adic analytic groups (we refer to [24, Interlude A] for a comprehensive list). Among these characterisations, in order to prove what could be regarded as *Hilbert's 5th problem for p-adic analytic groups*, Lazard himself proved that compact p-adic analytic groups are precisely the *profinite groups that are virtually pro-p groups of finite rank*– these are the profinite groups which contain a pro-p group of finite index such that all the subgroups of that pro-p group are finitely generated, and such that the necessary number of generators is bounded. The theory of p-adic analytic groups has since evolved into a rich area, and a number of exciting properties have been established: all of them, with the exception of the group $\mathbb{Z}_p$, satisfy *Golod-Shafarevich inequality* (Lubotzky [52]), they have *polynomial subgroup growth* (Lubotzky and Mann [53]), they are *verbally elliptic* (Jaikin-Zapirain [44]), etc.

Furthermore, if one starts with a general topological ring R and define analytic groups by analogy, the concept of p-adic analytic group is generalised to that of analytic group over R, which hereinafter will be referred to as R-*analytic group*. Thereby, Bourbaki (or better said, the bourbaquists) [11] and Serre [68] studied analytic groups over the local field $\mathbb{F}_p((t))$, i.e., the positive characteristic counterpart of p-adic analytic groups. Moreover, the second edition of the celebrated book *Analytic pro-p groups* [24] was provided with a further chapter concerning analytic groups over general *pro-p domains* and thus took the first steps of this broader theory. We recall that a pro-p domain is a *local Noetherian integral domain R which is complete with respect to the metric defined by the maximal ideal and whose residue field is finite of characteristic p* (Section 1.1 is devoted to exploring these rings and the concepts required in their definition). These more general groups possess interesting algebraic properties (see [14], [42], [43], [45] and [54]), albeit not as those enjoyed by the p-adic analytic groups. Worse, there is no characterisation, even at a conjectural level, of R-analytic groups purely in group-theoretic terms.

This thesis aims to develop further the theory mentioned above. Its objectives are twofold: on the one hand, to advance in the systematic study of analytic groups over general pro-p domains, which constitutes a somewhat belated sequel to [24, Chapter 13]; and, on the other hand, to provide this theory with new research results. Those mainly generalise known properties for p-adic to the broader context of R-analytic groups. We point out that, moreover, the p-adic case is usually a fundamental ingredient of our proofs.

We now outline the contents in greater detail: **Chapter 1** is an introduction to R-analytic groups taking [24, Chapter 13] as a starting point. We will pay special attention to standard groups, which perhaps constitute the main example of R-analytic groups, as well as to the Lie algebra associated with them. In view of the fact that many elementary concepts concerning analytic groups had still to be developed, we shall establish the machinery we will use throughout. For instance, in [24, p. 349], the authors highlighted that "*for more general analytic groups of the present chapter, such concepts [of submanifold and quotient manifold] would* *need to be developed*", which is precisely what we try to do in Section 1.5.

Chapter 2 is about linearity. Specifically we show that *when R is a pro-p domain of characteristic zero, every compact R-analytic group is linear*, i.e., it can be embedded in the eneral linear group $\mathrm{GL}_n(K)$ for a suitable field K. This partially answers a question from Lubotzky and Shalev (see Question 2 in page 311 of [54]). The proof we shall give is neat and based on the linearity of p-adic analytic groups. Besides, it has a model-theoretic flavour, as we embed the group in question in a convenient ultrapower of $\mathrm{GL}_n(\mathbb{Z}_p)$. Since model theory is not a central topic of this thesis, the proof is written, as far as possible, in such a way that prior knowledge is not required.

Chapter 3 is devoted to the Hausdorff dimension in compact R-analytic groups. We shall show that in a compact R-analytic group G, there exists a metric that encodes its analytic structure, and we will recall how to define the Hausdorff dimension corresponding to that metric, namely a fractal dimension $\mathrm{hdim}\colon \mathcal{P}(G) \to [0,1]$. The chapter consists of two main parts. Firstly, we shall study the relationship between the analytic and the Hausdorff dimensions of a closed submanifold. This study is based on the article [27] by Fernández-Alcober, Giannelli and González-Sánchez. Secondly, we will focus mainly on the case $R = \mathbb{F}_p[[t]]$, and describe the Hausdorff spectrum of compact $\mathbb{F}_p[[t]]$-analytic groups, namely the set

$$\mathrm{hspec}(G) = \{\mathrm{hdim}(H) \mid H \leq G \text{ is closed}\}.$$

Chapter 4 is concerned with words. A word is nothing but an element $w = w(x_1, \ldots, x_k)$ of the free group $F(x_1, \ldots, x_k)$ in k-generators; and given a group G, it naturally defines a map $w\colon G^{(k)} \to G$, which sends $(g_1, \ldots, g_k)$ to the element of G obtained by substituting g_i for x_i in w. In the setting of analytic groups, we will study some problems regarding words that were originally proposed by P. Hall [33]. In keeping with his terminology, we will prove that *in a compact R-analytic group every word is concise*, i.e., *whenever* $\mathrm{im}\, w$, *the image of the map w in G, is finite, the verbal subgroup $w(G) = \langle \mathrm{im}\, w \rangle$ is also finite.*

Finally, **Appendix A** contains a proof of Ado's Theorem for Lie algebras over principal ideal domains that are additionally free modules, since we will need this version of the theorem in Chapter 2.

At the end of each chapter, there is a section of *Notes*, where we detail the author's original contributions, or we make various comments.

This manuscript intends to be stand-alone, and accordingly, most concepts are thoroughly introduced. However, familiarity with profinite and pro-p groups is assumed, and if necessary the reader is referred to [24, Chapter 1].

Las frases de efecto [...] son una plaga maligna. Empezar por el principio, como si ese principio fuese la punta siempre visible de un hilo mal enrollado del que basta tirar y seguir tirando para llegar a la otra punta [...], como si [...] hubiésemos tenido en las manos un hilo liso y continuo del que no ha sido preciso dehacer nudos ni desenredar marañas.

(José Saramago, *La caverna*)

1
R-analytic groups

This thesis deals with analytic groups over pro-p domains. We shall dedicate this initial chapter to exhaustively presenting those groups and developing their theory, which will be essential throughout the subsequent chapters.

1.1 Pro-p domains

We start by presenting the coefficient rings for our manifolds.

A *local ring* is a ring R with a unique maximal ideal $\mathfrak{m}$, which for shortness we shall sometimes refer to as $(R, \mathfrak{m})$, and its *residue field* is the quotient field $R/\mathfrak{m}$. If $(R, \mathfrak{m})$ is a Noetherian local integral domain, according to the Krull intersection theorem (see, for instance, [3, Corollary 10.18]), the powers of the maximal ideal $\left\{\mathfrak{m}^N\right\}_{N\in\mathbb{N}}$ define a filtration series of R, that is, $\bigcap_{N\in\mathbb{N}} \mathfrak{m}^N = \{0\}$. As a consequence, R can be endowed with a topological structure. This topology, the so-called $\mathfrak{m}$-*adic topology*, is given by the norm $||\cdot||\colon R \to \mathbb{R}_{\geq 0}$ defined as $||0|| = 0$ and

$$||x|| = c^{-N} \text{ where } x \in \mathfrak{m}^N \setminus \mathfrak{m}^{N+1},$$

being $c > 1$ our favorite real number. The name comes from the fact that in this topology a neighbourhood basis of 0 is given by $\left\{\mathfrak{m}^N\right\}_{N\in\mathbb{N}}$.

Moreover, observe that the preceding norm is non-archimedean, that is, it satisfies the *strong triangle inequality*

$$||x + y|| \leq \max\{||x||, ||y||\},$$

and thus R is an *ultrametric space*. Consequently, if R is complete, any series $\sum_{N \in \mathbb{N}} a_N$ of elements in R is convergent provided that satisfying $a_N \in \mathfrak{m}^N$ (see [68, Part II, Chapter I, Theorem in pg. 64]).

In view of all this, we will chiefly work with the following family of rings:

Definition 1.1. A *pro-p domain* is a Noetherian local integral domain $(R, \mathfrak{m})$ which is complete with respect to the $\mathfrak{m}$-adic topology and whose residue field is finite of characteristic p.

Remark. The inverse limit $\varprojlim_{N \in \mathbb{N}} R/\mathfrak{m}^N$ is the completion of the local ring $(R, \mathfrak{m})$ with respect to the $\mathfrak{m}$-adic topology (see [32, Proposition 2.15]). When R is a pro-p domain, this completion coincides with R, and thus, since the residue field is finite of order a power of p, R is a pro-p group with respect to the addition.

The principal examples of those domains are the ring of p-adic integers $\mathbb{Z}_p$, or power series rings with coefficients in either $\mathbb{Z}_p$ or the finite field of p^c elements $\mathbb{F}_{p^c}$. Moreover, by virtue of the following classical theorem all the pro-p domains are structurally close to those above-mentioned examples.

Theorem 1.2 (Cohen's Structure Theorem [21])**.** *Let R be a pro-p domain of Krull dimension m. Then R contains a subring S such that $S \cong \mathbb{Z}_p[[t_1, \ldots, t_{m-1}]]$ if $\operatorname{char} R = 0$, or $S \cong \mathbb{F}_p[[t_1, \ldots, t_m]]$ if $\operatorname{char} R = p$ is positive. Moreover, R is a finitely generated S-module.*

In particular, since both $\mathbb{Z}_p$ and $\mathbb{F}_p[[t]]$ are *principal ideal domains (PID)* and R is a domain:

Corollary 1.3. *Let R be a pro-p domain of Krull dimension one. Then R is a finitely generated free $\mathbb{Z}_p$-module if $\operatorname{char}(R) = 0$ or a finitely generated free $\mathbb{F}_p[[t]]$-module if $\operatorname{char}(R) = p$ is positive.*

Hereinafter R will be a pro-p domain with maximal ideal $\mathfrak{m}$, and by default all the topological concepts will refer to the $\mathfrak{m}$-adic topology. Moreover, we will normally use K for its *fraction field* $\operatorname{Frac}(R)$.

In addition, the previous norm can be extended to $K = \mathrm{Frac}(R)$. Formally,

$$\left|\left|\frac{x}{y}\right|\right|_K := \frac{||x||_R}{||y||_R},$$

for all $x, y \in R$ such that $y \neq 0$. However, unless R is a PID, the topology induced by $||\cdot||_K$ does not extend the $\mathfrak{m}$-adic topology in R, in the sense that $\mathfrak{m}^N$ is not an open subset. If $(R, \mathfrak{m})$ is a PID, the set $k + \mathfrak{m}^N$, with $k \in K$ and $N \in \mathbb{N}$, is open with respect to the natural topology in K. Furthermore, if π a *uniformiser*, i.e. a generator of the maximal ideal $\mathfrak{m}$, for every element $k \in K$ there exists $N \in \mathbb{N}_0$ such that $\pi^N k \in R \setminus \mathfrak{m}$, and accordingly the norm in K can be alternatively defined as

$$||k||_K = c^N$$

(this definition does not depend on the uniformiser, considering that if π and ρ are two uniformisers of R, then there exists a unit $u \in \mathcal{U}(R)$ such that $\rho = u\pi$).

The distinction between PIDs and pro-p domains that are not PIDs will be fundamental in what follows.

1.1.1 Power series rings over pro-p domains

We denote by $Q[[X_1, \ldots, X_n]]$ the ring of formal power series in n variables $X_1, \ldots, X_n$ and with coefficients in the ring Q, i.e. its elements are the formal power series

$$\mathbf{F}(X_1, \ldots, X_n) = \sum_{\alpha_i \in \mathbb{N}_0} a_{\alpha_1, \ldots, \alpha_n} X_1^{\alpha_1} \ldots X_n^{\alpha_n},$$

where $a_{\alpha_1, \ldots, \alpha_n} \in Q$. We should also introduce some notation: the grading $|\alpha| = \sum_{i=1}^n \alpha_i \in \mathbb{N}_0$ for the n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^{(n)}$; and the customary abbreviations a_α for $a_{\alpha_1, \ldots, \alpha_n}$, $\mathbf{X}^\alpha$ for the monomial $X_1^{\alpha_1} \ldots X_n^{\alpha_n}$, $F(\mathbf{X})$ for $F(X_1, \ldots, X_n)$ and $Q[[\mathbf{X}]]$ for $Q[[X_1, \ldots, X_n]]$.

Considering that a pro-p domain R is a topological ring, we can analyse the convergence of formal power series, that is to say, a formal power series $F(\mathbf{X}) \in K[[\mathbf{X}]]$ with coefficients in the fraction field $K = \mathrm{Frac}(R)$ is convergent at $\mathbf{x} = (x_1, \ldots, x_n) \in R^{(n)}$ if $\sum_\alpha a_\alpha \mathbf{x}^\alpha$ is a convergent series of elements in R. Thereby, whenever we write $F(\mathbf{x})$, we implicitly mean that F is convergent at $\mathbf{x}$.

Lemma 1.4. *Let R be pro-p domain and $F(\boldsymbol{X}) \in R[[X_1, \ldots, X_n]]$. If $\boldsymbol{x} \in \mathfrak{m}^{(n)}$, then F is convergent at $\boldsymbol{x}$.*

More generally, let R be a pro-p domain, and for the sake of brevity let Λ be $\mathrm{Frac}(R)$ if R is a PID and R otherwise. According to [24, Lemma 6.45], the set of power series of $\Lambda[[X_1,\dots,X_n]]$ that are convergent in some open neighbourhood $\left(\mathfrak{m}^M\right)^{(n)}$ of 0 is given by

$$\Lambda_0[[\mathbf{X}]] := \left\{ \sum_{\alpha\in\mathbb{N}_0^{(n)}} a_\alpha \mathbf{X}^\alpha \in \Lambda[[\mathbf{X}]] \,\middle|\, \exists N \in \mathbb{N} \text{ such that } a_\alpha \mathfrak{m}^{N|\alpha|} \subseteq R \;\forall \alpha \neq \mathbf{0} \right\}. \tag{1.1}$$

Note that when R is not a PID, $\Lambda_0[[\mathbf{X}]]$ is simply $R[[\mathbf{X}]]$. Finally, we shall recall some well-known facts about those series. The reader could consult [10, Chapter III], [24, Section 6.6] or [68, Part II, Chapter II] for a more comprehensive background on power series.

Lemma 1.5 (cf. [24, Lemma 13.3])**.** *Let $F \in \Lambda_0[[X_1,\dots,X_n]]$ be a formal power series that is convergent in $\left(\mathfrak{m}^N\right)^{(n)}$. The evaluation map $F\colon \left(\mathfrak{m}^N\right)^{(n)} \to R$ is continuous.*

Theorem 1.6 (Universal property of power series rings, cf. [10, Chapter III, §5, Proposition 6])**.** *Let $\varphi\colon Q \to P$ be a ring homomorphism between normed topological rings. Suppose that P is complete and linearly topologised, and let $p_1,\dots,p_n \in P$ be* topologically nilpotent *elements, i.e.* $\lim_{j\to\infty} ||p_i^j||_P = 0$ *for all* $i \in \{1,\dots,n\}$. *Then, there exists a unique ring homomorphism*

$$\Phi_{\varphi,p_1,\dots,p_n}\colon Q[[X_1,\dots,X_n]] \to P$$

such that $\Phi_{\varphi,p_1,\dots,p_n}(X_i) = p_i$ for all $i \in \{1,\dots,n\}$ and $\Phi_{\varphi,p_1,\dots,p_n}(q) = \varphi(q)$ for all $q \in Q$. Moreover, if $\varphi\colon Q \to P$ is a continuous ring homomorphism between pro-p domains, $\Phi_{\varphi,p_1,\dots,p_n}$ is also continuous.

Observe that by the definition of the norm, if $(R,\mathfrak{m})$ is a pro-p domain, any element $x \in \mathfrak{m}$ is actually topologically nilpotent.

Lemma 1.7 (cf. [24, Corollary 6.48])**.** *Let $\boldsymbol{F} \in R[[X_1,\dots,X_n]]^{(m)}$ and $\mathbf{G} \in R[[X_1,\dots,X_m]]^{(l)}$ such that $\boldsymbol{F}(\mathbf{0}) \in \mathfrak{m}^{(m)}$, then the composition $\mathbf{G}\circ\boldsymbol{F}$ is well-defined and it belongs to $R[[X_1,\dots,X_n]]^{(l)}$.*

Proof. Let $\mathbf{G} = (G_1,\dots,G_l)$ and $\mathbf{Y}$ an m-tuple of indeterminates. Observe that $\mathbf{G}\circ\mathbf{F}$ is nothing but $(\Phi(G_1(\mathbf{Y})),\dots,\Phi(G_l(\mathbf{Y})))$, where $\Phi\colon R[[Y_1,\dots,Y_m]] \to R[[X_1,\dots,X_n]]$ is defined by the universal property as $\Phi(r) = r$ for all $r \in R$ and $\Phi(Y_i) = F_i(X_1,\dots,X_n) \in (\mathfrak{m}, X_1,\dots,X_n)$ for all $i \in \{1,\dots,m\}$. □

Finally, the following result will be fundamental when working with power series:

Lemma 1.8 (cf. [42, Lemma 9] and [68, Lemma in pg. 68])**.** *Let* $(R, \mathfrak{m})$ *be a pro-p domain,* $U \subseteq_o \mathfrak{m}^{(n)}$ *an open set and* $F \in \Lambda_0[[X_1, \ldots, X_n]]$ *such that* $F(\boldsymbol{x}) = 0$ *for all* $\boldsymbol{x} \in U$*. Then* $F = 0$*.*

Proof. It suffices to prove it for the case $n = 1$, so let $F(X) = \sum_{i \geq 0} a_i X^i$.

Suppose firstly that $F(X) \in R[[X]]$ and that $0 \in U$. We proceed by contraposition, that is, assume that $F(X) \neq 0$ and let m be the minimal integer such that $a_m \neq 0$. Define $G(X) = \sum_{i \geq m} a_i X^{i-m}$, then $G(0) = a_m \neq 0$, so by continuity there exists an open neighbourhood $V \subseteq U$ such that G does not vanish there. Moreover, since R is reduced, X^m does not vanish in $V \setminus \{0\}$, and therefore, since R is an integral domain,

$$F(X) = X^m \sum_{i \geq m} a_i X^{i-m} = X^m G(X)$$

does not vanish at the non-empty subset $V \setminus \{0\} \subseteq U$, contradicting the initial assumption.

Secondly, assume that R is a PID and $F(X) \in K[[X]]$, but $0 \in U$. Let π be an uniformiser of R. There exists $N \in \mathbb{N}$ such that $\mathfrak{m}^N \subseteq U$ and

$$\overline{a}_i := a_i \pi^{iN} \in R$$

for all $i \geq 0$. Since $\overline{F}(X) := \sum_{i \geq 0} \overline{a}_i X^i$ is convergent in $\mathfrak{m}^N$ and it vanishes there, from the first case we obtain that $\overline{F}(X) = 0$, i.e. $a_i \pi^{iN} = 0$ for all $i \geq 0$. Since R is an integral domain, $a_i = 0$ for all $i \geq 0$.

Finally, let $\Lambda = K$ if R is a PID and $\Lambda = R$ otherwise. Consider $x \in U$ and using the universal property define the ring isomorphism $\Phi\colon \Lambda[[X]] \to \Lambda[[X]]$ by $X \mapsto X + x$ and $\Phi|_\Lambda = \mathrm{Id}_\Lambda$. Since $\Phi(F)$ vanishes at $-x + U := \{-x + u \mid u \in U\}$ and this set contains 0, by the previous cases $\Phi(F) = 0$; so we conclude that $F = 0$ as Φ is a ring isomorphism. □

1.2 R-analytic groups

In this section we present the principal object of study of this thesis, namely R-analytic groups.

Definition 1.9. Let $U \subseteq_o R^{(n)}$. A function $f\colon U \to R$ is *R-analytic* at $x \in U$ if there exists an integer $N \in \mathbb{N}$ and a formal power series $F \in \Lambda_0[[X_1, \ldots, X_n]]$ such that

(i) $x + \left(\mathfrak{m}^N\right)^{(n)} \subseteq U$ and

(ii) $f(x+y) = F(y)$ for all $y \in \left(\mathfrak{m}^N\right)^{(n)}$.

Moreover, $f = (f_1, \ldots, f_m)\colon U \to R^{(m)}$ is R-analytic at x if each f_i is R-analytic at x, and $f = (f_1, \ldots, f_m)\colon U \to R^{(m)}$ is R-analytic at U when it is R-analytic at all the points of U.

Note that in accordance with the previous definition, f is analytic at any point $z \in x + \left(\mathfrak{m}^N\right)$. Indeed,

$$f(z+y) = f(x+y+(z-x)) = F(y+(z-x)) \ \forall y \in \left(\mathfrak{m}^N\right)^{(d)},$$

and $F(X+(z-x))$ is a convergent power series, since it can be regarded as $\Phi(F)$ where Φ is the ring isomorphism defined by virtue of the universal property by the assignation $X \mapsto X + (z-x)$.

Definition 1.10. Let R be a pro-p domain and M a topological space.

(i) An *R-chart* of M is a triple (U, ϕ, n) where $U \subseteq_o M$, $n \in \mathbb{N}_0$ and ϕ is a homeomorphism $\phi\colon U \to \phi(U)$ onto an open subset of $R^{(n)}$ with the subspace topology.

(ii) An *atlas* of M is a collection of R-charts $\mathcal{A} = \{(U_i, \phi_i, n_i)\}_{i \in I}$ such that $M = \cup_{i \in I} U_i$ and all the charts are *pairwise compatible*, that is, the *coordinate change maps* $\phi_i \circ \phi_j^{-1}|_{\phi_j(U_i \cap U_j)}\colon \phi_j(U_i \cap U_j) \to R^{(n_i)}$ are R-analytic maps for all $i, j \in I$.

Two atlases $\mathcal{A}$ and $\mathcal{B}$ are *compatible* when $\mathcal{A} \cup \mathcal{B}$ is also an atlas of M, and accordingly a *maximal atlas* is an atlas that contains any other compatible atlas.

(iii) An *R-analytic manifold* is a topological space endowed with an R-analytic structure defined by a maximal R-analytic atlas.

Remark 1.11. (i) We shall use the terminology "*(U, ϕ, n) is an R-chart of $x \in M$*" to mean that $x \in U$.
(ii) An R-chart (U, ϕ, n) is called *regular* if $\phi(U) = x + \left(\mathfrak{m}^N\right)^{(n)}$ for some $x \in R^{(n)}$ and $N \in \mathbb{N}$. This is just a technical assumption, considering that by restricting to a smaller open subset we can assume that any R-chart is regular.
(iii) The integer n is called the *dimension* of the R-chart (U, ϕ, n). After developing a little extra machinery in Corollary 1.30 we will show that if (U_1, ϕ_1, n_1) and

(U_2, ϕ_2, n_2) are two R-charts such that $U_1 \cap U_2 \neq \varnothing$, then $n_1 = n_2$. Therefore, the *dimension of* M at x is well-defined as the common dimension of the R-charts of x, and it will be denoted by $\dim_x M$.

It is worth remarking that over pro-p domains, unlike in real or complex manifolds, the dimension is not a topological property, but an analytic property determined by the chart. For example, $\mathbb{Z}_p$ and $\mathbb{Z}_p^{(2)}$ are isomorphic to each other as topological groups.

(iv) An R-analytic manifold is said to be *pure* when $\dim_x M$ is constant for all $x \in M$.

Mimicking Definition 1.9, we can define R-analytic maps between R-analytic manifolds:

Definition 1.12. Let M and N be R-analytic manifolds. A function $F\colon N \to M$ is *R-analytic* at $x \in N$ if there exists an R-chart (U, ϕ, n) of x in N and an R-chart (V, ψ, m) of $F(x)$ in M such that $F^{-1}(V)$ is open in N and

$$\psi \circ F \circ \phi^{-1}|_{\phi(U \cap F^{-1}(V))} \colon \phi\left(U \cap F^{-1}(V)\right) \to R^{(m)} \tag{1.2}$$

is an R-analytic map (according to Definition 1.9 (i)). Similarly, F is an *R-analytic map* when it is R-analytic at all the points of N.

Moreover, it is habitual to call the map (1.2) by F *in coordinates*, and on occasions we will use this term informally without specifying the R-charts we work with. Furthermore, we duly refer to F as *strictly analytic* at $S \subseteq N$, if in Definition 1.12 we can take the R-charts such that $S \subseteq U \cap F^{-1}(V)$ and there exists $\mathbf{H} \in R[[X_1, \ldots, X_n]]^{(m)}$ such that

$$\psi \circ F \circ \phi^{-1}(\phi(x)) = \mathbf{H}(x) \ \forall x \in S.$$

Adopting a term used by Serre [68] we will also define the following:

Definition 1.13. Let M be an R-analytic manifold, $U \subseteq_o M$ and $x \in U$. A family of R-analytic functions $\mathcal{F} = \{f_i \colon U \to R\}_{i=1}^n$ is a *coordinate system* of M at x, if there exits $U' \subseteq_o U$ such that $x \in U'$ and $(U', F|_{U'})$, where $F = (f_1, \ldots, f_n)$, is an R-chart.

Observe from the definition that whenever $\mathcal{F}$ is a coordinate system at x, then it is so locally around x.

Examples 1.14. (i) The canonical example of an R-analytic manifold is $M = \left(\mathfrak{m}^N\right)^{(n)}$ where $N, n \in \mathbb{N}$. The *canonical coordinate system* is $\{\pi_i\}_{i=1}^n$ where π_i is the ith projection map $\pi_i\colon \left(\mathfrak{m}^N\right)^{(n)} \to \mathfrak{m}^N$. Besides, for every $x \in M$, the components of the *translation map* $t_x\colon M \to M$, $y \mapsto y+x$, i.e. $\{\pi_i \circ t_x\}_{i=1}^n$, are also a coordinate system.

(ii) Let $K = \operatorname{Frac}(R)$, and endow $K^{(n)}$ with the topology given by the neighbourhood basis $\left\{k + \left(\mathfrak{m}^N\right)^{(n)}\right\}_{N\in\mathbb{N}}$ of $k \in K^{(n)}$. Then $K^{(n)}$ is an R-analytic manifold with respect to the atlas $\{(U_k, \varphi_k, n)\}_{k\in K^{(n)}}$ where $U_k = k + R^{(n)}$ and $\varphi_k\colon U_k \to R^{(n)}$, $x \mapsto x - k$ (recall that the topology in $K^{(n)}$ we are imposing does not in general coincide with the natural topology on the fraction field; in truth it never does unless R is a PID).

(iii) The set of matrices $\mathrm{M}_{n\times m}(\mathfrak{m})$, which can be naturally identified with $\mathfrak{m}^{(nm)}$, is clearly an R-analytic manifold, and so is the *general linear group* $\mathrm{GL}_n(R)$ with respect to the atlas $\{(U_A, \phi_A, n^2)\}_{A\in\mathrm{GL}_n(R)}$, where $U_A = A + \mathrm{M}_n(\mathfrak{m})$ and $\phi_A\colon U_A \to \mathrm{M}_n(\mathfrak{m})$, $A + M \mapsto M$.

(iv) Let M and N be R-analytic manifolds, with atlases $\{(U_i, \phi_i, n_i)\}_{i\in I}$ and $\{(V_j, \psi_j, m_j)\}_{j\in J}$. The direct product $M \times N$ is an R-analytic manifold with respect to the atlas $\{(U_i \times V_j, \phi_i \times \psi_j, n_i + m_j)\}_{i\in I,\ j\in J}$.

The following elementary properties can be easily deduced from Lemmata 1.5 and 1.7.

Lemma 1.15. (i) *Every R-analytic map is continuous.*

(ii) (cf. [24, Lemma 13.4]) *The composition of two R-analytic maps is an R-analytic map.*

(iii) *The composition of two strictly R-analytic maps is a strictly R-analytic map.*

We finish with the main definition:

Definition 1.16. An *R-analytic group* is a topological group G that is an R-analytic manifold such that

(i) the *multiplication map* $m\colon G \times G \to G$, $(g, h) \mapsto g \cdot h$ and

(ii) the *inversion map* $\iota\colon G \to G$, $g \mapsto g^{-1}$

are R-analytic maps.

Particularly, $\mathbb{Z}_p$-analytic groups are the foremost example of these groups, as well as the germ of the above definition. In the literature these groups have been referred to as *p-adic analytic groups*.

1.3 R-STANDARD GROUPS

The R-standard groups are a noteworthy family of R-analytic groups.

Definition 1.17. An *R-standard group* of level N and dimension d is an R-analytic group S with a global chart $\{(S, \phi, d)\}$ such that

(i) $\phi(S) = \left(\mathfrak{m}^N\right)^{(d)}$,

(ii) $\phi(1) = \mathbf{0}$ and

(iii) for all $j \in \{1, \ldots, d\}$ there exists a formal power series $F_j \in R[[X_1, \ldots, X_{2d}]]$ such that

$$\phi(xy) = (F_1(\phi(x), \phi(y)), \ldots, F_d(\phi(x), \phi(y))) \ \forall x, y \in S.$$

Any tuple of power series $\mathbf{F} = (F_1, \ldots, F_d)$ satisfying condition (iii) of the above definition must certainly also satisfy

(F1) $\mathbf{F}(\mathbf{X}, \mathbf{0}) = \mathbf{X}$ and $\mathbf{F}(\mathbf{0}, \mathbf{Y}) = \mathbf{Y}$ (in particular, each F_i has constant term equal to zero), and

(F2) $\mathbf{F}(\mathbf{X}, \mathbf{F}(\mathbf{Y}, \mathbf{Z})) = \mathbf{F}(\mathbf{F}(\mathbf{X}, \mathbf{Y}), \mathbf{Z})$,

(here $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are d-tuples of variables) as they are straightforward consequences of the fact that $\phi(1) = \mathbf{0}$ and the associativity of the group law. Conversely, any tuple $\mathbf{F} \in R[[X_1, \ldots, X_{2d}]]^{(d)}$ that satisfies the preceding two conditions endows $\left(\mathfrak{m}^N\right)^{(d)}$, for any $N \in \mathbb{N}$, with an R-standard group structure. Accordingly, a tuple of power series satisfying (F1) and (F2) is said to be a d-dimensional *formal group law*, and there exists a *formal inverse* of it, namely a tuple of power series $\mathbf{I} = (I_1, \ldots, I_d) \in R[[X_1, \ldots, X_d]]^{(d)}$ such that

$$\mathbf{F}(\mathbf{I}(\mathbf{X}), \mathbf{X}) = \mathbf{F}(\mathbf{X}, \mathbf{I}(\mathbf{X})) = \mathbf{0}$$

(see [24, Proposition 13.16 (ii)]). Sometimes we will denote an R-standard group by (S, ϕ) or $(S, \mathbf{F})$ to emphasise the rôle of the homeomorphism or the formal group law.

From (F1) we can further conclude that

$$\mathbf{F}(\mathbf{X},\mathbf{Y}) = \mathbf{X} + \mathbf{Y} + \mathbf{B}(\mathbf{X},\mathbf{Y}) + \mathbf{G}(\mathbf{X},\mathbf{Y}), \tag{1.3}$$

where $\mathbf{B}$ is bilinear and where all the monomials involved in $\mathbf{G}$ have total degree at least 3. Moreover, every monomial involved in $\mathbf{B}$ and $\mathbf{G}$ contains a non-zero power of X_i and Y_j for some $i, j \in \{1, \ldots, d\}$.

Remark 1.18 (cf. [68, Part II, Chapter IV, §7]). Starting from (1.3) we can obtain similar expressions for the formal inverse and the conjugation maps. In effect, it is a routine exercise to verify that if $\mathbf{I}$ is the formal inverse of $\mathbf{F}$ in (1.3), then

$$\mathbf{I}(\mathbf{X}) = -\mathbf{X} + \mathbf{B}(\mathbf{X},\mathbf{X}) + \tilde{\mathbf{G}}(\mathbf{X}), \tag{1.4}$$

where every monomial involved in $\tilde{\mathbf{G}}(\mathbf{X})$ has total degree at least 3; and consequently, the conjugation map has the next form in coordinates:

$$\mathbf{F}(\mathbf{I}(\mathbf{Y}),\mathbf{F}(\mathbf{X},\mathbf{Y})) = \mathbf{X} + \mathbf{B}(\mathbf{X},\mathbf{Y}) - \mathbf{B}(\mathbf{Y},\mathbf{X}) + \hat{\mathbf{G}}(\mathbf{X},\mathbf{Y}), \tag{1.5}$$

where every monomial involved in $\hat{\mathbf{G}}$ has total degree at least 3 and it contains a non-zero power of X_i and Y_j for some $i, j \in \{1, \ldots, d\}$.

Examples 1.19. (i) The additive group $\left(\mathfrak{m}^N\right)^{(d)}$ is an R-standard group with the *additive formal group law* $\mathbf{F}(\mathbf{X},\mathbf{Y}) = \mathbf{X} + \mathbf{Y}$.

(ii) The multiplicative R-standard group $G = 1 + \mathfrak{m}^N$, which is so with respect to the global chart $\phi\colon G \to \mathfrak{m}^N$, $1 + m \mapsto m$ and the *multiplicative formal group law* $F(X,Y) = X + Y + XY$.

(iii) We can generalise the previous two formal group laws: it is easy to verify that any 1-dimensional polynomial formal group law has the form $F_c(X,Y) = X + Y + cXY$ for some $c \in R$ (cf. [9, Corollary 2.2.4]).

(iv) Let $\mathrm{GL}_n^1(R)$ be the kernel of the modulo $\mathfrak{m}$ reduction map $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/\mathfrak{m})$, that is, $\mathrm{GL}_n^1(R) = I_n + \mathrm{M}_n(\mathfrak{m})$. This group is R-standard with the n^2-dimensional R-chart given by the assignation $I_n + A \mapsto A$ and the formal group law $\mathbf{F}(\mathbf{X},\mathbf{Y}) = \mathbf{X} + \mathbf{Y} + \mathbf{X} \cdot \mathbf{Y}$, where $\mathbf{X}$ and $\mathbf{Y}$ stand for n-by-n matrices of indeterminates.

(v) In positive characteristic p, we have the 2-dimensional formal group law

$$\mathbf{F}(X_1, X_2, Y_1, Y_2) = (X_1 + Y_1, X_2 + Y_2 + X_1^p Y_2),$$

which is attributed to Chevalley (cf. [18, Chapter II, §10, Example V]).

Given an R-standard group (S, ϕ) we can define the *R-standard filtration series* by

$$S_n := \phi^{-1}\left(\left(\mathfrak{m}^{N+n}\right)^{(d)}\right) \ \forall n \in \mathbb{N}_0, \tag{1.6}$$

where N and d stand for the level and the dimension of S. It readily follows from (1.5) that S_n is an open normal subgroup of S, for every $n \in \mathbb{N}$. Moreover, since R is compact, S is a compact topological group, so S_n has finite index in S. We can specify better:

Lemma 1.20 (cf. [27, Lemma 2.3])**.** *Let (S, ϕ, d) be an R-standard group of level N. Then,*

$$|S : S_n| = \left|\left(\mathfrak{m}^N\right)^{(d)} : \left(\mathfrak{m}^{N+n}\right)^{(d)}\right|,$$

where the latter stands for the index as additive groups.

Proof. From (1.3),

$$\phi(x) = \phi\left(xy^{-1}y\right) = \mathbf{F}\left(\phi\left(xy^{-1}\right), \phi(y)\right) = \phi(xy^{-1}) + \phi(y) + \mathbf{H}\left(\phi\left(xy^{-1}\right), \phi(y)\right),$$

where all the monomials involved in $\mathbf{H}(\mathbf{X}, \mathbf{Y})$ have total degree at least 2 and contain a non-zero power of X_i and Y_j for some $i, j \in \{1, \dots, d\}$.

Thus, if $\phi(x) - \phi(y) \in \left(\mathfrak{m}^K\right)^{(d)} \setminus \left(\mathfrak{m}^{K+1}\right)^{(d)}$, then

$$\phi\left(xy^{-1}\right) + \mathbf{H}\left(\phi\left(xy^{-1}\right), \phi(y)\right) = \phi(x) - \phi(y) \in \left(\mathfrak{m}^K\right)^{(d)} \setminus \left(\mathfrak{m}^{K+1}\right)^{(d)},$$

and so $\phi\left(xy^{-1}\right) \in \left(\mathfrak{m}^K\right)^{(d)} \setminus \left(\mathfrak{m}^{K+1}\right)^{(d)}$. Conversely, if $\phi\left(xy^{-1}\right) \in \left(\mathfrak{m}^K\right)^{(d)} \setminus \left(\mathfrak{m}^{K+1}\right)^{(d)}$, then

$$\phi(x) - \phi(y) \equiv \phi\left(xy^{-1}\right) \mod \left(\mathfrak{m}^{K+1}\right)^{(d)},$$

and so $\phi(x) - \phi(y) \in \left(\mathfrak{m}^K\right)^{(d)} \setminus \left(\mathfrak{m}^{K+1}\right)^{(d)}$.

In other words,

$$xy^{-1} \in S_n \iff \phi(x) - \phi(y) \in \left(\mathfrak{m}^{N+n}\right)^{(d)}. \qquad \square$$

Therefore, since $\log_{|R/\mathfrak{m}|} |\mathfrak{m}^i : \mathfrak{m}^{i+1}| = \dim_{R/\mathfrak{m}} \left(\mathfrak{m}^i/\mathfrak{m}^{i+1}\right)$, we have that

$$|S : S_n| = p^{cd \sum_{i=0}^{n-1} \dim_{R/\mathfrak{m}}(\mathfrak{m}^{N+i}/\mathfrak{m}^{N+i+1})}, \tag{1.7}$$

where p^c is the size of the residue field $R/\mathfrak{m}$, and particularly, S/S_n is a finite p-group. Forasmuch as S is a compact topological group with an open neighbourhood system of the identity $\{S_n\}_{n\in\mathbb{N}}$ such that S/S_n is a finite p-group for all $n \in \mathbb{N}$, we conclude that any R-standard group is actually a countably based pro-p group.

By the next result the study of R-analytic groups can be reduced, to some extent, to R-standard groups.

Lemma 1.21 (cf. [24, Theorem 13.20]). *Let G be an R-analytic group and (U, ϕ, d) an R-chart of the identity. Then, U contains an open R-standard subgroup of dimension $\dim_1 G$. In particular, every R-analytic group contains an open R-standard subgroup.*

Proof. We can assume that $\phi(1) = \mathbf{0}$ by composing with a convenient translation. Since the multiplication map m and the inversion map ι are R-analytic at 1, there exists $N \in \mathbb{N}$ such that $\left(\mathfrak{m}^N\right)^{(d)} \subseteq \phi(U)$ and some power series $F_j \in \Lambda_0[[X_1, \ldots, X_{2d}]]$ and $I_j \in \Lambda_0[[X_1, \ldots, X_d]]$, $j \in \{1, \ldots, d\}$, such that

$$\phi \circ m \circ (\phi, \phi)^{-1}(x, y) = (F_1(x, y), \ldots, F_d(x, y)) \;\; \forall x, y \in \left(\mathfrak{m}^N\right)^{(d)}$$

and

$$\phi \circ \iota \circ \phi^{-1}(x) = (I_1(x), \ldots, I_d(x)) \quad \forall x \in \left(\mathfrak{m}^N\right)^{(d)}$$

(actually, it is sufficient to consider the multiplication, since in $\left(\mathfrak{m}^N\right)^{(d)}$ the tuple of power series $\mathbf{I}$ is nothing but the formal inverse of $\mathbf{F}$).

In particular, if $\mathbf{F} = (F_1, \ldots, F_d)$ and $\mathbf{I} = (I_1, \ldots, I_d)$, then

$$\mathbf{0} = \phi(1) = \mathbf{F}(\phi(1), \phi(1)) = \mathbf{F}(\mathbf{0}, \mathbf{0}),$$

and

$$\mathbf{0} = \phi(1) = \mathbf{I}(\phi(1)) = \mathbf{I}(\mathbf{0}),$$

so each power series F_j and I_j has constant term equal to zero. Therefore, $\left(\mathfrak{m}^N\right)^{(d)}$ is closed with respect to both $\mathbf{F}$ and $\mathbf{I}$. Thus,

$$H := \phi^{-1}\left(\left(\mathfrak{m}^N\right)^{(d)}\right) \subseteq U$$

is an open subgroup of G such that

$$\phi(xy) = (F_1(\phi(x), \phi(y)), \dots, F_d(\phi(x), \phi(y))) \ \forall x, y \in H.$$

When R is not a PID, $\Lambda_0[[\mathbf{X}]] = R[[\mathbf{X}]]$ so $(H, \phi|_H)$ is an R-standard group of level N, dimension d and formal group law $\mathbf{F}$. Suppose now that R is a PID with uniformiser π and fraction field K, according to (1.3),

$$F_j(\mathbf{X}, \mathbf{Y}) = \mathbf{X} + \mathbf{Y} + \sum_{\substack{\alpha,\beta \in \mathbb{N}_0^{(d)} \setminus \{\mathbf{0}\} \\ |\alpha|+|\beta| \geq 2}} a_{j,\alpha,\beta} \mathbf{X}^\alpha \mathbf{Y}^\beta \in K[[\mathbf{X}, \mathbf{Y}]].$$

Since $\mathbf{F}$ is convergent in $\left(\mathfrak{m}^N\right)^{(d)}$, there exists $L \in \mathbb{N}_0$ such that $a_{j,\alpha,\beta}\pi^{L(|\alpha|+|\beta|)} \in R$. Define the power series

$$\overline{F}_j(\mathbf{X}, \mathbf{Y}) := \pi^{-L} F_j\left(\pi^L \mathbf{X}, \pi^L \mathbf{Y}\right) = \sum_{\alpha,\beta \in \mathbb{N}_0^{(d)} \setminus \{\mathbf{0}\}} \pi^{-L} a_{j,\alpha,\beta} \pi^{L(|\alpha|+|\beta|)} \mathbf{X}^\alpha \mathbf{Y}^\beta,$$

which is a power series with coefficients in R. Consequently,

$$\bar{H} := \phi^{-1}\left(\left(\mathfrak{m}^{N+L}\right)^{(d)}\right)$$

is an open subgroup of G, which is an R-standard group with the R-chart $\psi \colon \bar{H} \to \left(\mathfrak{m}^N\right)^{(d)}$, $h \mapsto \pi^{-L}\phi(h)$ and the formal group law $\overline{\mathbf{F}} = (\overline{F}_1, \dots, \overline{F}_d)$. Indeed,

$$\begin{aligned} \psi_j(xy) &= \pi^{-L}\phi_j(xy) = \pi^{-L} F_j(\phi(x), \phi(y)) \\ &= \overline{F}_j\left(\pi^{-L}\phi(x), \pi^{-L}\phi(y)\right) = \overline{F}_j(\psi(x), \psi(y)), \ \forall x, y \in \bar{H}. \end{aligned} \tag{1.8}$$

□

An open R-standard group (S, ϕ, d) can be used to obtain a natural atlas of G. Indeed, consider $\{(xS, \phi_x, d)\}_{x \in G}$, where $\phi_x \colon xS \to \left(\mathfrak{m}^N\right)^{(d)}$ is defined by $\phi_x(y) = \phi(x^{-1}y)$. Those R-charts are compatible, as

$$\phi_x \circ \phi_y^{-1} = \phi \circ L_{x^{-1}} \circ L_y \circ \phi^{-1} = \phi \circ L_{x^{-1}y} \circ \phi^{-1}$$

and $L_{x^{-1}y}$ is R-analytic. Moreover, this atlas is compatible with the initial R-analytic structure of G.

As a by-product we observe that

Corollary 1.22. *Let G be an R-analytic group. Then $\dim_x G$ is constant for all $x \in G$.*

Proof. We have indicated in Remark 1.11, although it will be proved in Corollary 1.30, that $\dim_x G$ is independent of the R-charts. By Lemma 1.21, there exists an open R-standard subgroup $S \leq G$ of dimension $d := \dim_1 G$, and the R-atlas $\{(xS, \phi_x, d)\}_{x \in G}$ shows that $\dim_x(G) = d$ for all $x \in G$. □

This common value is referred to as the *(analytic) dimension* of an R-analytic group, and on account of it, we will write the R-charts simply as the pair (U, ϕ).

1.3.1 R-standard groups and group operations

By virtue of Lemma 1.21 every R-analytic group contains an open pro-p subgroup, so compact R-analytic groups are profinite groups. Furthermore, assuming compactness Lemma 1.21 can be strengthened:

Lemma 1.23. *Let R be a pro-p domain that is not a PID. A compact R-analytic group G contains an open normal R-standard subgroup S such that for all $g \in G$ the conjugation map $c_g \colon S \to S$ is strictly R-analytic.*

Proof. Let G be a compact R-analytic group of dimension d. By Lemma 1.21, there exists a finite index R-standard subgroup (H, ϕ) of dimension d, level N, formal group law $\mathbf{F}$ and formal inverse $\mathbf{I}$. Let T be a left transversal for H in G. Since the conjugation maps are R-analytic at 1, for each $t \in T$ there exists an integer $N_t \geq N$ and some power series $C_j^t \in R[[X_1, \ldots, X_d]]$, $j \in \{1, \ldots, d\}$, such that

$$\phi\left(x^t\right) = \left(C_1^t(\phi(x)), \ldots, C_d^t(\phi(x))\right) \ \forall x \in \phi^{-1}\left(\left(\mathfrak{m}^{N_t}\right)^{(d)}\right).$$

Let $L = \max_{t \in T} N_t$, since $\left(\mathfrak{m}^L\right)^{(d)}$ is closed with respect to the tuples of power series $\mathbf{F}$ and $\mathbf{I}$, then $S := \phi^{-1}\left(\left(\mathfrak{m}^L\right)^{(d)}\right)$ is an open R-standard subgroup. Moreover,

$$C_j^t(\mathbf{0}) = C_j^t(\phi(1)) = 0,$$

so each C_j^t has constant term equal to zero, and thus, S is closed with respect to the conjugation by every $t \in T$ and $c_t \colon S \to S$ is strictly analytic in S for every $t \in T$. In addition, whenever $h \in H$ and $x \in S$ then

$$\phi\left(x^h\right) = \mathbf{F}(\mathbf{I}(\phi(h)), \mathbf{F}(\phi(x), \phi(h))),$$

so S is closed with respect to conjugating by h and $c_h \colon S \to S$ is strictly analytic in S. Since every element $g \in G$ can be written as th where $t \in T$ and $h \in H$, then S is closed with respect to the conjugation with g –i.e. S is normal in G– and $c_g = c_h \circ c_t$ is strictly analytic in S. □

For the remainder of the section, we keep the notation used throughout the previous proof and we recover the atlas associated to (S,ϕ), accordingly c_g is given in coordinates by the tuple of power series $\mathbf{C}_g=(C_1^g,\ldots,C_d^g)$ of the proof, and we can give an explicit description of the group operations in G:

Lemma 1.24. *Let G be an R-analytic group and let S be a open normal R-standard subgroup such that for all $g\in G$ the conjugation map $c_g\colon S\to S$ is strictly analytic. Suppose that with respect to the R-atlas induced by S the map c_g is given in coordinates by the tuple of power series $\boldsymbol{C}_g$. Let $t,r\in G$.*

(i) *The inverse in tS is given in coordinates by the tuple of power series $\boldsymbol{C}_{t^{-1}}\circ\boldsymbol{I}$. That is,*

$$\phi_{t^{-1}}\left(x^{-1}\right)=\left(\boldsymbol{C}_{t^{-1}}\circ\boldsymbol{I}\right)(\phi_t(x))\ \forall x\in tS.$$

(ii) *The multiplication in $tS\times rS$ is given in coordinates by the tuple of power series $\boldsymbol{F}(\boldsymbol{C}_r(\boldsymbol{X}),\,\boldsymbol{Y})$. That is,*

$$\phi_{tr}(xy)=\boldsymbol{F}(\boldsymbol{C}_r(\phi_t(x)),\phi_r(y))\ \forall x\in tS,y\in rS.$$

Proof. (i) Take $x=t\overline{x}\in tS$, then

$$\mathbf{C}_{t^{-1}}(\mathbf{I}(\phi_t(x)))=\mathbf{C}_{t^{-1}}\left(\phi\left(\overline{x}^{-1}\right)\right)=\phi\left(\left(\overline{x}^{-1}\right)^{t^{-1}}\right)=\phi_{t^{-1}}\left(x^{-1}\right).$$

(ii) Take $x=t\overline{x}\in tS$ and $y=r\overline{y}\in rS$, then

$$\phi_{tr}(xy)=\phi(\overline{x}^r\overline{y})=\mathbf{F}(\mathbf{C}_r(\phi(\overline{x})),\phi(\overline{y}))=\mathbf{F}(\mathbf{C}_r(\phi_t(x)),\phi_r(y)).\qquad\square$$

Finally, if we fix a left transversal T for S in G, we can work solely with the atlas $\{(tS,\phi_t)\}_{t\in T}$. In fact, for $x,y\in G$ such that $xS=yS$, let $A_x^y\colon\left(\mathfrak{m}^N\right)^{(d)}\to\left(\mathfrak{m}^N\right)^{(d)}$ be the R-analytic homeomorphism $\phi_y\circ\phi_x^{-1}$. Since $\phi_y=A_x^y\circ\phi_x$, Lemma 1.24(i) is restated as

$$\phi_r\left(x^{-1}\right)=\left(A_{t^{-1}}^r\circ\mathbf{C}_{t^{-1}}\circ\mathbf{I}\right)(\phi_t(x))\ \forall x\in tS,$$

whenever $rS=t^{-1}S$. Ditto multiplication, i.e.

$$\phi_p(xy)=A_{tr}^p\left(\mathbf{F}(\mathbf{C}_r(\phi_t(x)),\phi_r(y))\right)\ \forall x\in tS,y\in rS,$$

whenever $trS=pS$.

1.4 Lie algebras

Any R-standard group is associated with a so-called Lie algebra. The objective of this section is to describe this construction by following [24, Section 13.3], as well as to reproduce for general pro-p domains the results in [68, Part II, Chapter III, § 10].

Let $(S, \mathbf{F})$ be an R-standard group of level N and dimension d and formal inverse $\mathbf{I}$. In accordance with the notation in (1.3) we can associate to $\mathbf{F}$ the Lie bracket

$$[\mathbf{X}, \mathbf{Y}]_{\mathbf{F}} := \mathbf{B}(\mathbf{X}, \mathbf{Y}) - \mathbf{B}(\mathbf{Y}, \mathbf{X}),$$

which will be simply denoted by $[\cdot, \cdot]$ when there is no risk of confusion. Let us verify that $[\cdot, \cdot]_{\mathbf{F}}$ is an actual formal *Lie bracket* (see Appendix A for the precise definition of Lie bracket). Obviously, it is bilinear and $[\mathbf{X}, \mathbf{X}] = 0$. Further it satisfies the Jacobi identity:

Lemma 1.25 (cf. [24, Lemma 13.24] and [68, Part II, Chapter IV, § 7.6])**.** *Let $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\mathbf{Z}$ be d-tuples of variables. Then,*

$$[\boldsymbol{X}, [\boldsymbol{Y}, \mathbf{Z}]] + [\boldsymbol{Y}, [\mathbf{Z}, \boldsymbol{X}]] + [\mathbf{Z}, [\boldsymbol{X}, \boldsymbol{Y}]] = \mathbf{0}.$$

Proof. The result follows from the so-called Hall-Witt identity (cf. [68, Part I, Proposition 1.1]), the non-commutative version of Jacobi's identity. Accordingly, every group G satisfies the identity:

$$[x^y, [y, z]]\,[y^z, [z, x]]\,[z^x, [x, y]] = 1. \tag{1.9}$$

Hereinafter, $O(n)$ stands for formal power series in two d-tuples of variables $\mathbf{X}$ and $\mathbf{Y}$, all whose monomials have degree at least n and contain a non-zero power of X_i and Y_j for some $i, j \in \{1, \ldots, d\}$. Moreover, the formal conjugation $\mathbf{F}(\mathbf{I}(\mathbf{Y}), \mathbf{F}(\mathbf{X}, \mathbf{Y}))$ will be abbreviated by $\mathbf{X}^{\mathbf{Y}}$, and if

$$\mathbf{C}(\mathbf{X}, \mathbf{Y}) := \mathbf{F}(\mathbf{I}(\mathbf{X}), \mathbf{F}(\mathbf{I}(\mathbf{Y}), \mathbf{F}(\mathbf{X}, \mathbf{Y})))$$

is the formal commutator, according to (1.3) - (1.5), we have that $\mathbf{X}^{\mathbf{Y}} = \mathbf{X} + O(2)$ and $[\mathbf{X}, \mathbf{Y}] = \mathbf{C}(\mathbf{X}, \mathbf{Y}) + O(4)$. Therefore,

$$[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] = \mathbf{C}\left(\mathbf{X}^{\mathbf{Y}}, \mathbf{C}(\mathbf{Y}, \mathbf{Z})\right) + O(4).$$

Thus,

$$\begin{aligned}\mathbf{0} &= \mathbf{F}\left(\mathbf{C}\left(\mathbf{X}^{\mathbf{Y}}, \mathbf{C}\left(\mathbf{Y}, \mathbf{Z}\right)\right), \mathbf{F}\left(\mathbf{C}\left(\mathbf{Y}^{\mathbf{Z}}, \mathbf{C}\left(\mathbf{Z}, \mathbf{X}\right)\right), \mathbf{C}\left(\mathbf{Z}^{\mathbf{X}}, \mathbf{C}\left(\mathbf{X}, \mathbf{Y}\right)\right)\right)\right) \\ &= [\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] + O(4),\end{aligned}$$

using (1.9) in the first equality. Hence, since $[\cdot,[\cdot,\cdot]]$ only involves monomials of total degree 3,

$$[\mathbf{X},[\mathbf{Y},\mathbf{Z}]]+[\mathbf{Y},[\mathbf{Z},\mathbf{X}]]+[\mathbf{Z},[\mathbf{X},\mathbf{Y}]]=\mathbf{0}. \qquad \square$$

Consequently, this Lie bracket makes $R^{(d)}$ into a *Lie algebra*, referred to as the *Lie algebra associated to S*.

The natural maps between R-Lie algebras are R-algebra homomorphisms, i.e. R-linear homomorphims that preserve Lie brackets. As customary, the linear approximation of an R-analytic map is given by its differential map. In fact, if $\mathbf{U}(\mathbf{X}) \in R[[X_1,\dots,X_n]]^{(m)}$ is a tuple of power series such that $\mathbf{U}(\mathbf{0})=\mathbf{0}$, let us denote by $\mathbf{U}^{[k]}$ the homogeneous part of degree k in $\mathbf{U}$. In particular, the linear term of $\mathbf{U}$ is

$$\mathbf{U}^{[1]}(\mathbf{X})=\left(\sum_{i=1}^{n} a_{1i}X_i,\dots,\sum_{i=1}^{n} a_{mi}X_i\right).$$

The *differential* of $\mathbf{U}$ is the R-linear map $D\mathbf{U}\colon R^{(n)}\to R^{(m)}$ defined by $\mathbf{U}^{[1]}$ and the *Jacobian matrix* of $\mathbf{U}$ is the matrix $\mathcal{J}\mathbf{U}:=(a_{ij})_{i,j}$ that defines the differential. It is a routine exercise to verify the so-called *chain rule*, that is, if $\mathbf{V}$ is an n-tuple of power series such that $\mathbf{V}(\mathbf{0})=\mathbf{0}$, then

$$D(\mathbf{U}\circ\mathbf{V})=D\mathbf{U}\circ D\mathbf{V} \text{ and } \mathcal{J}(\mathbf{U}\circ\mathbf{V})=\mathcal{J}\mathbf{U}\cdot\mathcal{J}\mathbf{V}$$

–the composition is well-defined in view of Lemma 1.7–.

Moreover, if $f\colon U\subseteq R^{(n)}\to R^{(m)}$ is an R-analytic map such that $f(x)=\mathbf{0}$, then, according to Definition 1.12, there exists a tuple of power series $\mathbf{U}$ such that $\mathbf{U}(y)=f(x+y)$ for all y in a suitable neighbourhood U of x, and, particularly, $\mathbf{U}(\mathbf{0})=\mathbf{0}$. The differential of f at x, denoted by D_xf, is defined as $D\mathbf{U}$ and the Jacobian matrix of f at x, denoted by $\mathcal{J}_xf$ is defined to be $\mathcal{J}\mathbf{U}$.

Let $(S,\mathbf{F},d)$ and $(T,\mathbf{G},e)$ be two R-standard groups, a *formal morphism* is a tuple of power series $\mathbf{U}\in R[[X_1,\dots,X_d]]^{(e)}$ such that

$$\mathbf{U}(\mathbf{F}(\mathbf{X},\mathbf{Y}))=\mathbf{G}(\mathbf{U}(\mathbf{X}),\mathbf{U}(\mathbf{Y})).$$

Let us prove that its differential respects Lie brackets:

Lemma 1.26 (cf. [24, Proposition 13.26] and [35, Section 14.2])**.** *Let* $\boldsymbol{U}$ *be a formal group morphism. The linear map* $D\boldsymbol{U}$ *is an R-Lie algebra homomorphism, that is,*

$$\mathbf{U}^{[1]}\left([\boldsymbol{X},\boldsymbol{Y}]_{\boldsymbol{F}}\right)=\left[\mathbf{U}^{[1]}(\boldsymbol{X}),\mathbf{U}^{[1]}(\boldsymbol{Y})\right]_{\mathbf{G}}.$$

Proof. We will briefly describe the proof. For simplicity we will use $\cdot$ and $\cdot^{-1}$ for the multiplication and inversion, instead of the corresponding formal group laws and inverses; we will use as well the customary notation $O(n)$. Since $\mathbf{U}(\mathbf{X}) = \mathbf{U}^{[1]}(\mathbf{X}) + O(2)$, then

$$\mathbf{U}(\mathbf{X})^{-1} \cdot \mathbf{U}(\mathbf{Y})^{-1} \cdot \mathbf{U}(\mathbf{X}) \cdot \mathbf{U}(\mathbf{Y}) = \left[\mathbf{U}^{[1]}(\mathbf{X}), \mathbf{U}^{[1]}(\mathbf{Y})\right]_{\mathbf{G}} + O(3)$$

and

$$\mathbf{U}\left(\mathbf{X}^{-1} \cdot \mathbf{Y}^{-1} \cdot \mathbf{X} \cdot \mathbf{Y}\right) = \mathbf{U}^{[1]}\left([\mathbf{X}, \mathbf{Y}]_{\mathbf{F}}\right) + O(3),$$

and this yields the result by comparing monomials of degree 2. □

Therefore, it turns out that although an R-standard group might have various global R-charts, and consequently various formal groups laws. However, whenever ϕ_1 and ϕ_2 are two global R-charts of the same R-standard group S, then $\phi_2 \circ \phi_1^{-1}$ is a formal group morphism and the corresponding associated Lie algebras are isomorphic via $D_{\mathbf{0}}\left(\phi_2 \circ \phi_1^{-1}\right)$. That is, the Lie algebra associated to S is unique up to isomorphism. Besides as the upshot of it all we formalise the idea of associating a Lie algebra to each R-standard group.

Theorem 1.27 (cf. [24, Section 13.3])**.** *The assignments* $(S, \mathbf{F}) \mapsto \left(R^{(d)}, [\cdot,\cdot]_{\boldsymbol{F}}\right)$ *and* $f \mapsto D_{\mathbf{0}}f$ *define a functor from the category* $\mathfrak{RStd}$ *consisting of R-standard groups* $(S, \boldsymbol{F})$ *as objects and formal group morphisms* f *as morphisms to the category* $\mathfrak{RLie}$ *consisting of Lie R-algebras as objects and R-Lie algebra homomorphisms as morphisms.*

Let now $F \colon N \to M$ be an R-analytic map, and let (U, ϕ, n) and (V, ψ, m) be R-charts at, respectively, $x \in N$ and $F(x) \in M$, assume further that $\phi(x) = \mathbf{0}$ and $\psi(F(x)) = \mathbf{0}$ (this is a technical assumption, since by composing with an adequate translation we can always obtain a valid homeomorphism). The *differential* and the *Jacobian matrix* of F at x (with respect to those charts), denoted by D_xF and $\mathcal{J}_xF$, are defined respectively as $D_{\mathbf{0}}\left(\psi \circ F \circ \phi^{-1}\right)$ and $\mathcal{J}_{\mathbf{0}}\left(\psi \circ F \circ \phi^{-1}\right)$. Those definitions are related with the classical analytic Jacobian. Indeed, whenever $\mathbf{U}(\mathbf{X}) \in R[X_1, \ldots, X_n]]^{(m)}$, its "classical" Jacobian matrix is

$$\operatorname{Jac}\mathbf{U}(X_1, \ldots, X_n) := \left(\partial_j U_i(X_1, \ldots, X_n)\right)_{i,j},$$

where ∂_j stands for the partial formal derivative with respect to the variable X_j. Therefore, if locally around $\mathbf{0}$ we have that $\psi \circ F \circ \phi^{-1}(z) = \mathbf{U}(z)$, then $\mathcal{J}_xF$ is nothing but $\operatorname{Jac}\mathbf{U}(\mathbf{0})$. Moreover, whenever $y \in U \cap F^{-1}(V)$ then $\overline{\phi}(z) :=$

$\phi(z) - \phi(y)$ and $\overline{\psi}(z) := \psi(z) - \psi(F(y))$ are R-charts of y and $F(y)$ such that $\overline{\phi}(y) = \mathbf{0}$ and $\overline{\psi}(F(y)) = \mathbf{0}$. Consequently, in view of the analytic chain rule,

$$\begin{aligned}\mathcal{J}_y F = \mathcal{J}_{\mathbf{0}}\left(\overline{\psi} \circ F \circ \overline{\phi}^{-1}\right) &= \mathcal{J}_{\mathbf{0}}\left(\mathbf{U}(\mathbf{X} + \phi(y)) - \psi(F(y))\right) \\ &= \operatorname{Jac}\mathbf{U}(\mathbf{X} + \phi(y))|_{\mathbf{X}=\mathbf{0}} = \operatorname{Jac}\mathbf{U}(\mathbf{X})|_{\mathbf{X}=\phi(y)}. \end{aligned} \tag{1.10}$$

In the light of this example and in order to simplify notation, from now on, we will assume that if (U, ϕ, n) is an R-chart at x, then $\phi(x) = \mathbf{0}$.

However, these preceding definitions do depend on the chosen R-charts. Nevertheless, we will be chiefly interested in the rank of the Jacobian. Firstly, we have the following result.

Lemma 1.28. *Let $\mathbf{U} = (U_1, \ldots, U_n) \in R[[X_1, \ldots, X_n]]^{(n)}$ be a tuple of power series with constant term equal to zero such that the linear part of U_i is $\sum_{j=1}^n a_{ij}X_j$. Suppose that $\mathbf{U}$ is invertible, that is, there exists a tuple of power series $\mathbf{V} = (V_1, \ldots, V_n) \in R[[X_1, \ldots, X_n]]^{(n)}$ such that*

$$V_i \circ \mathbf{U}(\boldsymbol{X}) = U_i \circ \mathbf{V}(\boldsymbol{X}) = X_i$$

for all $i \in \{1, \ldots, n\}$. Then $(a_{ij})_{i,j} \in \mathrm{GL}_n(R)$.

Proof. If the linear part of V_i is $\sum_{j=1}^n b_{ij}X_j$ then

$$(\mathbf{U} \circ \mathbf{V})(\mathbf{X}) = \left(\sum_{j=1}^n \sum_{k=1}^n a_{1k}b_{kj}X_j + O(2), \ldots, \sum_{j=1}^n \sum_{k=1}^n a_{nk}b_{kj}X_j + O(2)\right) = \mathbf{X},$$

where the abbreviation $O(n)$ stands for a collection of power series in which every monomial has total degree of at least n. Therefore, $(b_{ij})_{i,j}$ is the inverse of $(a_{ij})_{i,j}$. □

Example 1.29. For instance, in an R-analytic group G the *left multiplication* map $L_g \colon G \to G$, $x \mapsto gx$ is analytic and so is its inverse $L_{g^{-1}}$. Thus, from Lemma 1.28, we obtain that $\mathcal{J}_x L_g \in \mathrm{GL}_n(R)$ at every point $x \in G$. Obviously, the same holds for right multiplication maps.

Corollary 1.30. *Let M be an R-analytic manifold and let $\{(U_i, \phi_i, d_i)\}_{i=1}^2$ be R-charts. If $U_1 \cap U_2 \neq \varnothing$, then $d_1 = d_2$.*

Proof. Let $x \in U_1 \cap U_2$. By definition, $\phi_1 \circ \phi_2^{-1}$ is an invertible R-analytic map, so in view of Lemma 1.28, $\mathcal{J}_x\left(\phi_1 \circ \phi_2^{-1}\right) \in \mathrm{M}_{d_1 \times d_2}(R)$ is invertible, so $d_1 = d_2$. □

Corollary 1.31. *Let* $F\colon N \to M$ *be an* R*-analytic map. Let* $\{(U_i, \phi_i, n)\}_{i=1}^2$ *and* $\{(V_i, \psi_i, m)\}_{i=1}^2$ *be* R*-charts at respectively* x *and* $F(x)$*. Then,*

$$\operatorname{rk} \mathcal{J}_{\mathbf{0}} \left(\psi_1 \circ F \circ \phi_1^{-1}\right) = \operatorname{rk} \mathcal{J}_{\mathbf{0}} \left(\psi_2 \circ F \circ \phi_2^{-1}\right).$$

Proof. Note that when restricted to $\phi_1\left(U_1 \cap U_2 \cap F^{-1}(V_1) \cap F^{-1}(V_2)\right)$ we have

$$\psi_1 \circ F \circ \phi_1^{-1} = \psi_1 \circ \psi_2^{-1} \circ \psi_2 \circ F \circ \phi_2^{-1} \circ \phi_2 \circ \phi_1^{-1},$$

so

$$\mathcal{J}_{\mathbf{0}} \left(\psi_1 \circ F \circ \phi_1^{-1}\right) = \mathcal{J}_{\mathbf{0}} \left(\psi_1 \circ \psi_2^{-1}\right) \mathcal{J}_{\mathbf{0}} \left(\psi_2 \circ F \circ \phi_2^{-1}\right) \mathcal{J}_{\mathbf{0}} \left(\phi_2 \circ \phi_1^{-1}\right).$$

Since $\psi_1 \circ \psi_2^{-1}$ and $\phi_1 \circ \phi_2^{-1}$ have R-analytic inverses, the results follows by Lemma 1.28. □

Therefore, even though $\mathcal{J}_x F$ and $D_x F$ are not unambiguously defined, expressions such as "$\mathcal{J}_x F \in \mathrm{GL}_n(R)$" or "$D_x F$ *is injective*" are mathematically precise, and so is the next definition:

Definition 1.32. Let $F\colon N \to M$ be an R-analytic map. Then,

(i) F is a *weak immersion* at x if $\operatorname{rk} \mathcal{J}_x F = \dim_x N$.

(ii) F is a *weak submersion* at x if $\operatorname{rk} \mathcal{J}_x F = \dim_x M$.

Similarly, F is a *weak immersion* if it is a weak immersion at all points of the domain, and *weak submersions* are defined likewise.

The following theorem will be important henceforward, and it can be proved by reproducing the proof of [24, Theorem 6.37].

Theorem 1.33 (Inverse Function Theorem)**.** *Let* R *be a ring and let* $U_1, \ldots, U_n \in R[[X_1, \ldots, X_n]]$ *be power series with constant term equal to zero. If the linear term of* U_i *is* $\sum_{j=1}^n a_{ij} X_j$*, assume that* $(a_{ij})_{i,j} \in \mathrm{GL}_n(R)$*. Then there exist* $V_1, \ldots, V_n \in R[[X_1, \ldots, X_n]]$ *with constant term equal to zero such that*

$$(V_i \circ \boldsymbol{U})(\boldsymbol{X}) = (U_i \circ \boldsymbol{V})(\boldsymbol{X}) = X_i \text{ for all } i \in \{1, \ldots, n\}.$$

Hence, the Jacobian matrix does provide information about the local invertibility of an analytic map.

Definition 1.34. An R-analytic map $F\colon N \to M$ is R*-bianalytic** at $x \in N$ when $\mathcal{J}_x F \in \mathrm{GL}_n(R)$. Moreover, F is R*-bianalytic* when it is bianalytic at all the points in N.

*In [68], Serre used the term *étale* for these morphisms.

Therefore, the concept of bianalytic map is the natural notion of isomorphism between R-analytic manifolds, since it is an R-analytic homeomorphism with an analytic inverse. The following result is straightforward from the definitions, but essential in the setting of R-analytic manifolds.

Theorem 1.35. *An R-analytic map $F\colon N \to M$ is R-bianalytic at $x \in N$ if and only if D_xF is an R-module isomorphism.*

All this can be used to analyse the change of coordinates.

Theorem 1.36 (Change of coordinates)**.** *Let M be an R-analytic manifold, $x \in M$, $n = \dim_x M$, $U \subseteq_o M$ such that $x \in U$ and $\{f_i\colon U \to R\}_{i=1}^n$ a family of R-analytic functions such that $f_i(x) = 0$ for all i. The following are equivalent:*

(i) *$\{f_i\}_{i=1}^n$ is a coordinate system at x.*

(ii) *$F = (f_1, \dots, f_n)$ is R-bianalytic at x.*

(iii) *$\{D_x f_i\}$ is a basis for $\left(R^{(n)}\right)^*$, the dual R-module of $R^{(n)}$.*

Proof. (i) $\Rightarrow$ (ii). There exists an open neighbourhood V of x such that $(V, F|_V, n)$ is an R-chart of x, so F is the identity in coordinates.
(ii) $\Rightarrow$ (i). Since $\mathcal{J}_xF \in GL_n(R)$, according to Theorem 1.35, F is locally an R-analytic homeomorphism, i.e. there exists an open set V such that $F\colon V \to F(V) \subseteq R^{(n)}$ is a homeomorphism. Moreover, (V, F, n) is easily seen to be compatible with the initial manifold structure of M.
(ii) $\Leftrightarrow$ (iii). Let $(b_{ij})_{i,j}$ be *any* Jacobian matrix of F at x. The collection of linear maps $\left\{\sum_{j=1}^n b_{1j}X_j, \dots, \sum_{j=1}^n b_{nj}X_j\right\}$ is a basis of $\left(R^{(n)}\right)^*$ if and only if $(b_{ij})_{i,j} \in \mathrm{GL}_n(R)$. □

As already commented in the introduction, the theory of analytical varieties has been widely developed over fields, chiefly on the complex field. Plenty of the results, however, depend on the vector space structure of the associated Lie algebras (also known as tangent spaces in Lie theory), and, consequently, this theory is based on results from linear algebra. Despite the tangents of R-analytic manifolds being R-modules, most of the reasonings can be replicated, albeit with certain additional conditions.

Our tangent spaces being R-modules, we should use the notion of *residue rank* instead of the usual rank. In fact, let Q be a ring and $A \in \mathrm{M}_{n\times m}(Q)$. The residue rank of A, denoted by $\mathrm{res.\,rk}(A)$, is the maximum $r \in \mathbb{N}_0$ such that there exists an $r \times r$ minor Δ_r of A satisfying $\Delta_r \in \mathcal{U}(Q)$. Obviously, when Q is a field, this notion coincides with the usual rank. When Q is a pro-p domain $(R, \mathfrak{m})$, then

$$\operatorname{res.rk}(A) = \operatorname{rk}_{R/\mathfrak{m}}\left(\overline{A}\right),$$

where $\overline{A}$ is the matrix obtained by reducing modulo $\mathfrak{m}$ the entries of A, and $\operatorname{rk}_{R/\mathfrak{m}}$ means that the rank is computed over the residue field $R/\mathfrak{m}$.

It is straightforward to see that Corollary 1.31 also holds, with the selfsame proof, if we substitute the residue rank by the rank, so the following definition is unambiguous.

Definition 1.37. Let $F\colon N \to M$ be an R-analytic map. Then,

(i) F is an *immersion* at x if $\operatorname{res.rk} \mathcal{J}_x F = \dim_x N$.

(ii) F is an *submersion* at x if $\operatorname{res.rk} \mathcal{J}_x F = \dim_x M$.

Similarly, an *immersion* is an R-analytic map that is an immersion at any point of the domain, and the same for *submersions.*

Lemma 1.38. *Let $F\colon N \to M$ be an R-analytic immersion. Then F looks in coordinates like the R-linear monomorphism*

$$\iota\colon R^{(n)} \to R^{(m)},\ (x_1, \ldots, x_n) \mapsto \left(x_1, \ldots, x_n, 0, \overset{(m-n)}{\ldots}, 0\right).$$

More concretely, for each $x \in N$ there exist an R-chart (U, ϕ, n) of x in N and an R-chart (V, ψ, m) of $F(x)$ in M such that the following diagram is commutative:

$$\begin{array}{ccc} U & \xrightarrow{\quad F \quad} & V \\ \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \psi} \\ R^{(n)} & \xrightarrow{\quad \iota \quad} & R^{(m)}. \end{array}$$

Similarly, a submersion looks in coordinates like the R-linear epimorphism

$$\pi\colon R^{(n)} \to R^{(m)},\ (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_m).$$

Proof. We shall only prove the result for the immersion, as it is proved in a similar fashion for the submersion. Firstly, since the question is local we can assume, by working in coordinates, that $x = \mathbf{0} \in U = \left(\mathfrak{m}^L\right)^{(n)} \subseteq_o N$, that $F(x) = \mathbf{0} \in V = \left(\mathfrak{m}^L\right)^{(m)} \subseteq_o M$ and that $F \in R[[X_1, \dots, X_n]]^{(m)}$.

Secondly, for simplicity, we can assume that the residue rank of the first n columns of $\mathcal{J}_{\mathbf{0}}F$ is n, that is, if $\tilde{F} = (F_1, \dots, F_n)$ then $\operatorname{res.rk} \mathcal{J}_{\mathbf{0}}\tilde{F} = n$.

Let $W = \left(\mathfrak{m}^L\right)^{(m-n)}$, and define the map $\Phi\colon N \times W \to M$, $(x, w) \mapsto F(x) + (\mathbf{0}, w)$. Then,

$$\mathcal{J}_{\mathbf{0}}\Phi = \left(\begin{array}{c|c} \mathcal{J}_{\mathbf{0}}F & \begin{matrix} \mathbf{0} \\ I_{m-n} \end{matrix} \end{array} \right) \in \mathrm{M}_m(R),$$

and so $\operatorname{res.rk} \mathcal{J}_{\mathbf{0}}\Phi = m$. By Theorem 1.33, there exists a local inverse of Φ, that is, there exist some open subsets U', V' and W' of, respectively, $R^{(n)}$, $R^{(m)}$ and $R^{(m-n)}$ such that $\Psi\colon V' \to U' \times W'$ is the inverse of $\Phi|_{U' \times W'}$. That is, the following diagram is commutative

$$\begin{array}{ccc} U' & \xrightarrow{\quad F \quad} & V' \\ \Big\downarrow{\scriptstyle \iota} & & \Big\downarrow{\scriptstyle \Psi} \\ U' \times \{0\}^{(m-n)} & \xrightarrow{\Psi \circ F \circ \iota^{-1}} & U' \times W', \end{array}$$

where $\iota(x) = (x, \mathbf{0})$, and

$$\Psi \circ F\colon U' \to U' \times W',\ (x_1, \dots, x_n) \mapsto \left(x_1, \dots, x_n, 0, \overset{(m-n)}{\dots}, 0\right),$$

as we were required. □

In particular, since an R-bianalytic map is both an immersion and submersion, it looks in coordinates as the identity.

Definition 1.39. An R-analytic map $F\colon N \to M$ is a *subimmersion* at x when there exist $x \in U \subseteq_o N$, $F(x) \in V \subseteq_o M$ and an R-analytic manifold W such that $F|_U$ is the composition

$$U \overset{\pi}{\to} W \overset{\iota}{\to} V,$$

where π is a submersion and ι is an immersion.

Lemma 1.40. *Let $F\colon N \to M$ be an R-analytic map. The following are equivalent:*

(i) *F is subimmersion at x.*

(ii) *F looks in coordinates around x like the R-linear homomorphism*

$$\overline{F}\colon R^{(n)} \to R^{(m)},\ (x_1,\dots,x_n) \mapsto (x_1,\dots,x_r,0,\dots,0)$$

for some $r \le \min\{n,m\}$, where $n = \dim_x N$ and $m = \dim_{F(x)} M$.

Proof. The proof follows from Lemma 1.38.

(i) $\Rightarrow$ (ii). By definition F has locally the form $U \xrightarrow{\pi} W \xrightarrow{\iota} V$. Let $r = \dim_{\pi(x)} W$. According to Lemma 1.38, π and ι look in coordinates as

$$\overline{\pi}\colon R^{(n)} \to R^{(r)},\ (x_1,\dots,x_n) \mapsto (x_1,\dots,x_r)$$

and

$$\overline{\iota}\colon R^{(r)} \to R^{(m)},\ (x_1,\dots,x_r) \mapsto (x_1,\dots,x_r,0,\overset{(m-r)}{\dots},0),$$

so F looks like their composition.
(ii) $\Rightarrow$ (i). There exist R-charts (U,ϕ,n) at x and (V,ψ,m) at $F(x)$ such that $\psi \circ F \circ \phi^{-1} = \overline{F} = \overline{\iota} \circ \overline{\pi}$ at $\phi(U)$, where $\overline{\pi}$ and $\overline{\iota}$ are defined as in the preceding implication. Hence $\overline{W} = \overline{\pi}\,(\phi(U))$ is an open subset of $R^{(r)}$ and the following diagram is commutative:

$$\begin{array}{ccccc} U & & \xrightarrow{\quad F \quad} & & V \\ \downarrow{\scriptstyle \phi} & & & & \uparrow{\scriptstyle \psi^{-1}} \\ \phi(U) & \xrightarrow{\overline{\pi}} & \overline{W} & \xrightarrow{\overline{\iota}} & \psi(V) \end{array}$$

The result follows since $\overline{\pi} \circ \phi$ is a submersion and $\psi^{-1} \circ \overline{\iota}$ is an immersion. $\square$

The following lemma is proved reproducing the arguments of [68, Theorem in pg. 86]. We will rely on the next basic result:

Lemma 1.41. *Let Q be a ring of characteristic zero and $F \in Q[[X_1,\dots,X_n]]$. Suppose that the formal derivatives on the last m variables are 0, that is, $\partial_j F = 0$ for all $j \in \{n-m+1,\dots,n\}$. Then $F \in Q[[X_1,\dots,X_{n-m}]]$.*

Proposition 1.42. *Let R be a pro-p domain of characteristic zero and let $F\colon N \to M$ be an R-analytic map. Suppose that there exists $r \in \mathbb{N}_0$ such that $\operatorname{rk} \mathcal{J}_y F = \operatorname{res.rk} \mathcal{J}_y F = r$ for all y in an open neighbourhood U of x. Then F is a subimmersion.*

Proof. Let $n = \dim_x N$ and $m = \dim_{F(x)} M$. Since the question is local, working in coordinates, we can assume that $x = \mathbf{0} \in U = \left(\mathfrak{m}^L\right)^{(n)} \subseteq_o N$, $F(x) = \mathbf{0} \in V = \left(\mathfrak{m}^L\right)^{(m)} \subseteq_o M$ and that $F \in R[[X_1, \ldots, X_n]]^{(m)}$. Assume for simplicity that if $\tilde{F} = (F_1, \ldots, F_r)$ then $\operatorname{res.rk} \mathcal{J}_{\mathbf{0}} \tilde{F} = r$. Thus,

$$\{F_1, \ldots, F_r, \pi_{r+1}, \ldots, \pi_n\}$$

is a coordinate system of N at U. Hence, if we consider U as $\left(\mathfrak{m}^L\right)^{(r)} \times \left(\mathfrak{m}^L\right)^{(n-r)}$, after a change of coordinates we can assume that $\tilde{F}(x_1, x_2) = x_1$, that is, then

$$F(x_1, x_2) = (x_1, \psi(x_1, x_2)),$$

for some $\psi \in R[[X_1, \ldots, X_n]]^{(m-r)}$ such that $\operatorname{res.rk} \mathcal{J}_{\mathbf{0}} \psi = 0$. We shall check up on ψ, whether it is independent of the second variable in a neighbourhood of 0.

First of all, $\partial_2 \psi = \mathbf{0}$ in U, where $\partial_2 \psi$ stands for the matrix of formal derivatives on the last $n - r$ variables. Otherwise, according to (1.10), there would be $y \in U$ such that $\operatorname{rk} \mathcal{J}_y F > r$, which is a contradiction. Therefore, since $\operatorname{char} R = 0$, from Lemma 1.41, ψ is independent of the last $n - r$ coordinates. That is,

$$F(x_1, x_2) = (x_1, \psi(x_1)).$$

Hence, if $\pi \colon \left(\mathfrak{m}^L\right)^{(n)} \to \left(\mathfrak{m}^L\right)^{(r)}$ is the projection onto the first r coordinates, then $F = (\operatorname{Id} \times \psi) \circ \pi$, and thus, F is locally the composition of a submersion and an immersion. □

1.5 Construction of manifolds

This section is devoted to constructing new analytic structures starting from an initial R-analytic manifold, both by changing the coefficient ring to a convenient subring of R or by developing concepts such as submanifolds or quotient manifolds.

1.5.1 Restriction of scalars

In this subsection we will illustrate how to induce a manifold structure over a suitable subring. For that we shall follow the procedure of [24, Example 13.6].

Let $(R, \mathfrak{m})$ be a pro-p domain and Q a subring that is itself a pro-p domain with maximal ideal $\mathfrak{n} := \mathfrak{m} \cap Q$. Suppose further that R is a finitely generated free Q-module. For instance, in view of Cohen's Structure Theorem whenever

$\dim_{\mathrm{Krull}}(R) = 1$, then R is a finitely generated free $\mathbb{Z}_p$-module if $\operatorname{char} R = 0$, or a finitely generated free $\mathbb{F}_p[[t]]$-module if $\operatorname{char} R = p$ is positive.

Let M be an R-analytic manifold and $\sigma\colon R \to Q^{(e)}$ a Q-module isomorphism, that is, if we fix a basis $\{v_1, \dots, v_e\}$ for R as a Q-module, then σ is

$$\sigma\left(\sum_{i=1}^{e} q_i v_i\right) = (q_1, \dots, q_e).$$

For each R-chart (U, ϕ, n) of M, we can define the triple $\left(U, \sigma^{(n)} \circ \phi, ne\right)$, which is actually a Q-chart. Indeed, we only have to check that $\left(\sigma^{(n)} \circ \phi\right)(U)$ is open in $Q^{(ne)}$. Firstly, note that if τ is the inverse of σ, then $\tau\left(\left(\mathfrak{n}^N\right)^{(e)}\right) \subseteq \mathfrak{m}^N$ for all $N \in \mathbb{N}$. Secondly, let $x \in \phi(U)$, since $\phi(U)$ is open in $R^{(n)}$, there exists $N \in \mathbb{N}$ such that

$$x + \left(\mathfrak{m}^N\right)^{(n)} \subseteq \phi(U),$$

and therefore,

$$\sigma^{(n)}(x) + \left(\mathfrak{n}^N\right)^{(ne)} \subseteq \sigma^{(n)}\left(x + \left(\mathfrak{m}^N\right)^{(n)}\right) \subseteq \left(\sigma^{(n)} \circ \phi\right)(U).$$

Moreover, the Q-charts $\left(U, \sigma^{(n)} \circ \phi, ne\right)$ and $\left(V, \sigma^{(n)} \circ \psi, ne\right)$ are compatible (note that the compatibility is a requirement only when $U \cap V \neq \varnothing$, so in view of Corollary 1.30, the charts must be of equal dimension). In fact,

$$\left(\sigma^{(n)} \circ \phi\right) \circ \left(\sigma^{(n)} \circ \psi\right)^{-1} = \sigma^{(n)} \circ \left(\phi \circ \psi^{-1}\right) \circ \left(\sigma^{-1}\right)^{(n)},$$

and since $\phi \circ \psi^{-1}$ is R-analytic, the result follows by the next lemma:

Lemma 1.43 (cf. [24, Exercise 13.4])**.** *Let $F \in \Lambda_0(R)[[X_1, \dots, X_n]]$, then*

$$\sigma \circ F \circ \left(\sigma^{-1}\right)^{(n)} : \mathfrak{n}^{(en)} \to Q^{(e)}$$

is strictly Q-analytic.

Proof. As customary $\Lambda(R)$ stands for the fraction field $\mathrm{Frac}(R)$ if R is PID and for R otherwise, and the same for $\Lambda(Q)$.

Observe that when R is PID, since R is an integral ring extension of Q, then $\dim_{\mathrm{Krull}}(Q) = \dim_{\mathrm{Krull}}(R) = 1$, so the pro-$p$ domain Q is also a discrete valuation ring. Therefore, if π and ρ are uniformisers of respectively R and Q, then $\rho = \pi^N$ for some $N \in \mathbb{N}$.

Let $\{v_1, \ldots, v_e\}$ be the basis for R as free Q-module that corresponds to σ and let us denote σ^{-1} by τ. Moreover, suppose that

$$F(X_1, \ldots, X_n) = \sum_{\alpha \in \mathbb{N}_0^{(n)}} a_\alpha X_1^{\alpha_1} \ldots X_n^{\alpha_n} \in \Lambda_0(R)[[X_1, \ldots, X_n]].$$

We will show that there exist some power series $F_l^* \in \Lambda_0(Q)[[X_1, \ldots, X_{en}]]$, $l \in \{1, \ldots, e\}$, such that

$$F \circ \tau^{(n)}(y_1, \ldots, y_{en}) = \sum_{l=1}^{e} v_l F_l^* (y_1, \ldots, y_{en}) \ \forall y_j \in \mathfrak{n}.$$

That is, whenever $x_j = \sum_{i=1}^{e} v_i y_{ij}$ for some $y_{ij} \in \mathfrak{n}$, $j \in \{1, \ldots, n\}$, then

$$F(x_1, \ldots, x_n) = \sum_{l=1}^{e} v_l F^*(y_{11}, \ldots, y_{en}).$$

First of all, there exist some elements there exist some elements $a_\alpha(k,l) \in \Lambda(Q)$ such that

$$a_\alpha v_k = \sum_{l=1}^{e} a_\alpha(k,l) v_l.$$

In order to prove the preceding when R is a PID, we should have taken into account that since $F \in \Lambda_0(R)[[\mathbf{X}]]$ and since $\rho = \pi^N$, there exists a big enough integer $L \in \mathbb{N}$ such that $a_\alpha \rho^{|\alpha| L} \in R$ for all α. In particular,

$$a_\alpha(k,l)\rho^{|\alpha| L} \in Q \ \forall k \in \{1, \ldots, e\}. \tag{1.11}$$

Furthermore, by an application of the multinomial theorem, for any tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^{(n)}$ there exist some elements $\gamma_k(\beta) \in Q$ such that

$$\prod_{j=1}^{n} \left(\sum_{i=1}^{e} v_i Y_{ij} \right)^{\alpha_j} = \sum_{|\beta|=|\alpha|} \sum_{k=1}^{e} \gamma_k(\beta) v_k \prod_{i=1}^{e} \prod_{j=1}^{n} Y_{ij}^{\beta_{ij}},$$

where the Y_{ij}'s are indeterminates. Finally, the desired power series are defined as

$$F_l^*(Y_{11}, \ldots, Y_{en}) = \sum_{\alpha \in \mathbb{N}_0^{(d)}} \sum_{|\beta|=|\alpha|} \sum_{k=1}^{e} a_\alpha(k,l) \gamma_k(\beta) \prod_{i=1}^{e} \prod_{j=1}^{n} Y_{ij}^{\beta_{ij}}.$$

When R is not a PID these power series are clearly in $Q[[\mathbf{X}]]$. In contrast, when R is a PID, $F_l^* \in \Lambda_0(Q)[[\mathbf{X}]]$ by virtue of (1.11). □

Furthermore, the preceding Q-manifold structure is independent of the isomorphism σ chosen, i.e. of the Q-basis of R chosen. Actually, it suffices to prove that for the R-analytic manifold $\mathfrak{m}^N$ any two Q-module isomorphisms σ and $\tilde{\sigma}$ give rise to equivalent Q-charts. For that note that

$$\tilde{\sigma} \circ \sigma^{-1} \colon \sigma\left(\mathfrak{m}^N\right) \to \tilde{\sigma}\left(\mathfrak{m}^N\right)$$

is nothing but the linear map defined by the change of basis matrix $A \in \mathrm{GL}_e(Q)$, and so, it is a Q-bianalytic map. Thus, we will refer to this procedure of generating new manifolds, simply as *restriction of scalars*, with no need of specifying the isomorphism. To finish, let us observe the following particular case:

Corollary 1.44. *Let R be a pro-p domain of Krull dimension one. Then any R-analytic group is by restriction of scalars either a p-adic analytic group if* $\mathrm{char}(R) = 0$, *or an $\mathbb{F}_p[[t]]$-analytic group if* $\mathrm{char}(R) = p$ *is positive.*

Under certain conditions the converse of this result is also true.

Theorem 1.45. *Let G be a non-discrete R-analytic group.*

(i) (cf. [24, Theorem 13.23]) *If G admits a p-adic analytic group structure, then R is a finitely generated $\mathbb{Z}_p$-module.*

(ii) (cf. [45, Theorem 1.1]) *Suppose that G is finitely generated (as topological group). If G admits an $\mathbb{F}_p[[t]]$-analytic group structure, then R is a finitely generated $\mathbb{F}_p[[t]]$-module.*

In the latter article, the authors already observed that the result does not hold for non finitely generated groups, on account of the additive topological groups $\mathbb{F}_p[[t_1]]$ and $\mathbb{F}_p[[t_1, t_2]]$ being isomorphic to one another. Nevertheless, whether for finitely generated groups the analytic structure determines the base ring is a sensible question, in this direction it is tempting to speculate that:

Conjecture 1.46. Let G be a non-discrete finitely generated topological group that admits both an R-analytic and a Q-analytic group structure. Then, R and Q share the Krull dimension and the characteristic.

1.5.2 Submanifolds

The concept of submanifold has to be developed when treating manifolds over general pro-p domains. The theory of analytic manifolds over local fields supplies various equivalent definitions (cf. [68, pg. 89]). Here we reproduce some of those, underscoring that any definition should reflect the fact that the set has R-analytic manifold structure by itself.

Definition 1.47. Let $F\colon N \to M$ be an injective weak immersion. Then $F(N)$ is an *immersed subset* of M.

For example, let $\mathfrak{m}$ be the maximal ideal of $\mathbb{Z}_p[[t]]$ with the natural manifold structure. If we endow the set $t\mathfrak{m}$ with the global chart $\psi\colon t\mathfrak{m} \to \mathfrak{m}$, $tx \mapsto x$, the inclusion $\iota\colon t\mathfrak{m} \to \mathfrak{m}$ is a weak immersion, as $\mathcal{J}_x\left(\mathrm{Id} \circ \iota \circ \psi^{-1}\right) = (t)$ for all $x \in t\mathfrak{m}$. However, in this example, the topology of $t\mathfrak{m}$, which is chosen purposely to make ψ a homeomorphism, does not coincide with the subspace topology. Since it is natural to ask for compatibility between topological structures, we define:

Definition 1.48. Let M be an R-analytic manifold. Then $S \subseteq M$ is an *R-analytic submanifold* when for each $s \in S$ there exist $k_s \in \mathbb{N}_0$, an open neighbourhood U_s of s in S and an R-chart (V_s, ϕ_s, d_s) of s in M such that

- $U_s \subseteq V_s$ and
- $\phi_s(U_s) = \phi_s(V_s) \cap \left(R^{(k_s)} \times \{0\}^{(d_s - k_s)}\right)$.

The integer k_s is the *dimension of S at s*, denoted by $\dim_s S$.

With the subspace topology S is an R-analytic manifold, as each point $s \in S$ can be endowed with the R-chart $(U_s, (\phi_{s,1}, \ldots, \phi_{s,k_s}), k_s)$.

An immediate application of Lemma 1.38 yields:

Proposition 1.49. *Let $F\colon N \to M$ an injective R-analytic map. Then $F(N)$ is an R-analytic submanifold of M if and only if F is an immersion.*

Remark 1.50. Note that R-analytic submanifolds are locally closed. Indeed, let M be an R-analytic manifold and $S \subseteq M$ a submanifold. For each $s \in S$ there exists an two subsets $U_s \subseteq_o S$ and $V_s \subseteq_o M$ containing s such that $\phi_s\left(U_s\right)$ is defined by some linear equations in $\phi(V_s)$. In particular, U_s is closed in V_s, that is, $V_s \setminus U_s$ is open in V_s. Hence, if $V = \cup_{s \in S} V_s$, then $V \setminus S = \bigcup_{s \in S} V_s \setminus U_s$ is open in V, so S is closed in the open set V.

Another straightforward observation yields:

Lemma 1.51. *Let M be an R-analytic manifold and S a submanifold. Suppose that $\dim_s M = \dim_s S$ for all $s \in S$. Then S is open in M.*

Proof. In keeping with the notation of Definition 1.48: since $k_s = d_s$ and ϕ is a homeomorphism, then $U_s = V_s$ and so $S = \bigcup_{s \in S} V_s$ is open in M. $\square$

Definition 1.48 strengthens the definition of R-analytic submanifold used in [27]:

Definition 1.52. Let M be an R-analytic manifold. Then $S \subseteq M$ is a *weak R-analytic submanifold* if for each $s \in S$ there exist an open neighbourhood U_s of s in S, an R-chart (V_s, ϕ_s, d_s) of s in M and a K-vector space $E_s \leq K^{(d_s)}$ ($K = \operatorname{Frac} R$) such that

- $U_s \subseteq V_s$ and
- $\phi_s(U_s) = \phi_s(V_s) \cap E_s$.

Over fields Definitions 1.48 and 1.52 are equivalent. Certainly, by a convenient change of basis we can assume that $E_s = K^{(k)} \times \{\mathbf{0}\}$, where $k = \dim_K(E_s)$. Nonetheless, as we shall illustrate with an example, in general they are not equal. Let $R = \mathbb{Z}_2[[t]]$ with maximal ideal $\mathfrak{m} = (2,t)R$, $K = \mathbb{Q}_2((t))$, the R-analytic manifold $M = \mathfrak{m}^{(2)}$ and the K-vector space:

$$E = \left\{(x,y) \in K^{(2)} \mid 2x - ty = 0\right\}.$$

Then, $M \cap E = \{(ta, 2a) \mid a \in R\}$ is clearly a weak R-analytic submanifold, and it is endowed with the global R-chart

$$\psi\colon M \cap E \to R, (ta, 2a) \mapsto a.$$

However, $\mathcal{J}_s\left(\mathrm{Id} \circ \iota \circ \psi^{-1}\right) = \begin{pmatrix} t \\ 2 \end{pmatrix}$ for all $s \in M \cap E$, and therefore ι is not an immersion but a weak immersion. In fact, the change of basis of $K^{(2)}$ from the canonical basis to $\beta = \{(t,2),(0,1)\}$ is the linear map $L_A\colon K^{(2)} \to K^{(2)}$ defined by the change of basis matrix

$$A = \begin{pmatrix} 1/t & 0 \\ -2/t & 1 \end{pmatrix},$$

and it maps E to $K^{(1)} \times \{0\}$. However, L_A is not an R-bianalytic map. This illustrates forby the difficulty to determine a valid coordinate system for a weak submanifold. Over fields, the natural way of doing so is as before, that is, by fixing a basis $\beta = \{v_1, \ldots, v_k\}$ for E_s, and considering the map

$$\psi\colon \phi_s(V_s) \cap E_s \to K^{(k)},\ \sum_{i=1}^{k} \alpha_i v_i \mapsto (\alpha_1, \ldots, \alpha_k) \in K^{(k)}.$$

However, over general pro-p domains depending on the choice of β, it might occur that $\operatorname{im}\psi$ is not contained in $R^{(k)}$, and so ψ might not be an R-chart.

In classical Lie theory, authors already distinguish between immersed (the equivalent of Definition 1.47) and embedded (the equivalent of Definition 1.48) submanifolds (see [51, Chapter 5]), the latter being a stronger condition. Nevertheless, when the manifold is compact both concepts coincide, and so do they when working with analytic manifolds over principal ideal pro-p domains (cf. [68, Part II, Section III.11.2]). Over general pro-p domains, however, it is apparent now that the notion of submanifold we work with must be categorically specified.

In [45, Section 4], $\mathbb{F}_q[[t]]$-analytic submanifolds are characterized as fibers of analytic maps. More precisely, they prove the following:

Proposition 1.53 (see [45, Corollary 4.2])**.** *Let M be an $\mathbb{F}_q[[t]]$-analytic manifold and a subset $S \subseteq M$. Suppose that*

(i) *S is homogeneous, i.e. it is contained in a single orbit of the action of the group of R-bianalytic automorphisms of the manifold M, and*

(ii) *S is an analytic subset, i.e. for each $s \in S$ there exist an open neighbourhood U and some $\mathbb{F}_q[[t]]$-analytic maps $\{f_i \colon U \to \mathbb{F}_q[[t]]\}_{i\in I}$ such that*

$$S \cap U = \{x \in U \mid f_i(x) = 0 \ \forall i \in I\}.$$

Then S is an $\mathbb{F}_q[[t]]$-analytic submanifold.†

Observe that when M is an R-analytic group, the action of the left multiplication maps is transitive, and thus in this situation, every subset is homogeneous.

In order to replicate a version of this for general pro-p domains, given an R-analytic manifold M and $S \subseteq M$, we say that S is an *R-analytic subset* of M when for every $s \in S$ there exist an open neighbourhood U of s and some R-analytic maps $\{f_i \colon U \to R \mid i = 1, \dots, r_s\}$ such that

$$S \cap U = \{y \in U \mid f_i(y) = 0 \ \forall i = 1, \dots, r_s\}.$$

In other words, an analytic subset is locally the nullset of some analytic functions. This definition obviously extends that of submanifold. Moreover, note that since $R[[X_1, \dots, X_n]]$ is Noetherian (see [49, Theorem IV.9.4]), the definition can be in principle relaxed allowing r_s to be infinite.

†This result is valid regardless of the definition of submanifold, as the base ring is a PID.

Definition 1.54. Let S be an R-analytic subset of an R-analytic manifold M. A point $s \in S$ is a *regular point* when there exists an open neighbourhood U of s and some R-analytic maps $\{f_i : U \to R \mid i = 1, \dots, r_s\}$ such that

(i) $S \cap U = \{y \in U \mid f_i(y) = 0 \ \forall i = 1, \dots, r_s\}$ and

(ii) if $F := (f_1, \dots, f_{r_s})$, then $\operatorname{res.rk}(\mathcal{J}_s F) = r_s$.

The integer r_s is termed the *corank* of S at s, and by Lemma 1.38, we know that $r_s \leq \dim_s S$.

Lemma 1.55. *Let M be an R-analytic manifold and $S \subseteq M$. Then, S is a submanifold if and only if all the points of S are regular points.*

Proof. The *only if* is immediate from the definition and Lemma 1.38. For the *if*, let $s \in S$ and $d = \dim_s S$, suppose that s is a regular point of corank r, then there exists an open neighbourhood U of s such that

$$S \cap U = \{y \in U \mid f_i(y) = 0 \ \forall i = 1, \dots, r\}.$$

If $F = (f_1, \dots, f_r)$, then $\operatorname{res.rk} \mathcal{J}_s F = r$, so we can extend F to a coordinate system $\{f_i\}_{i=1}^{d}$ of s in M, and with respect to that coordinate system for a possibly smaller open set $U' \subseteq U$ we have

$$S \cap U' = \{y \in U' \mid f_1(y) = \cdots = f_r(y) = 0\},$$

and thus M is a submanifold of dimension $d - r$ at s. □

Additionally, in R-analytic groups, we can blend the notions of analytic and group substructure. Accordingly,

Definition 1.56. Let G be an R-analytic group. An *analytic subgroup* is a subgroup $H \leq G$ that is besides an R-analytic submanifold.

Lemma 1.57. *An R-analytic subgroup is closed.*

Proof. It is a general fact that in a topological group a locally closed subgroup is actually closed, so the result is straightforward from Remark 1.50. To make it explicit, H being locally closed means that H is open in its topological closure $\overline{H}$. Suppose by contradiction that $H \neq \overline{H}$, so that there exists a non-trivial left coset $gH \subseteq \overline{H}$. Since H and gH are open dense subsets of $\overline{H}$, then $gH \cap H \neq \varnothing$, contradicting the disjointness of the cosets. □

1.5.3 Analytic quotients

An account of quotients of analytic groups over PIDs might be found in [68, Part II, Section III.12]. In parallel, [11, Chapter III, §1.6] proves that the quotient of an $\mathbb{F}_p[[t]]$-analytic group by an analytic subgroup is analytic. The proof of the general case follows by classical arguments.

Lemma 1.58. *Let G be an R-analytic group of dimension d, let $N \trianglelefteq G$ be a normal analytic subgroup of dimension k and let $\mathrm{pr}\colon \mathfrak{m}^{(d)} \to \mathfrak{m}^{(d-k)}$ be the projection onto the last $d-k$ coordinates. Then, there exists an R-chart (V, ψ) of 1 such that $\mathrm{pr} \circ \psi(x) = \mathrm{pr} \circ \psi(y)$ if and only if $xy^{-1} \in N$.*

Notation. An R-chart (V, ψ) as before is said to be *adapted* to N.

Proof. Firstly, we will fix some notation: let $r = d - k$ and let $\mathrm{pr}_1\colon \mathfrak{m}^{(d)} \to \mathfrak{m}^{(k)}$ and $\mathrm{pr}_2\colon \mathfrak{m}^{(d)} \to \mathfrak{m}^{(r)}$ be the projections onto respectively the first k and the last r coordinates, i.e. $\mathrm{pr}_2 = \mathrm{pr}$ according to the notation of the statement. Since N is a submanifold, there exists an R-chart (U, ϕ) at 1 in G such that

$$N \cap U = \{x \in U \mid \phi_{k+1}(x) = \cdots = \phi_d(x) = 0\}.$$

Moreover, by Lemma 1.21 we can assume that U is a d-dimensional R-standard group that via ϕ can be identified with $\left(\mathfrak{m}^M\right)^{(d)}$ for some $M \in \mathbb{N}$, such that the group multiplication is given by the formal group law $\mathbf{F}$ and the identity is $\mathbf{0}$. Therefore,

$$N \cap U = \{y \in U \mid y_{k+1} = \cdots = y_d = 0\},$$

and so $(N \cap U, \phi_1)$, where $\phi_1 = \mathrm{pr}_1|_U$, is an R-chart for $\mathbf{0}$ in N. Additionally, define the r-dimensional submanifold

$$W = \{y \in U \mid y_1 = \cdots = y_k = 0\},$$

with the R-chart (W, ϕ_2), where $\phi_2 = \mathrm{pr}_2|_U$, and consider also the R-analytic map:

$$\begin{aligned} F\colon (N \cap U) \times W &\longrightarrow U \\ (n, w) &\longmapsto n \cdot w = \mathbf{F}(n, w). \end{aligned}$$

Note that $\mathcal{J}_{\mathbf{0}}\left(F \circ (\phi_1, \phi_2)^{-1}\right) = I_d$, so by the Inverse Function Theorem F is locally an R-bianalytic map, i.e. there exist open neighbourhoods U_1 of $\mathbf{0}$

in N, U_2 of $\mathbf{0}$ in W and U_3 of $\mathbf{0}$ in U such that $F\colon U_1 \times U_2 \to U_3$ is an R-bianalytic map. Furthermore, by taking smaller neighbourhoods, we can assume that $U_1 = \left(\mathfrak{m}^L\right)^{(d)} \cap N$ for some $L \geq M$, and, in particular, $U_1 \leq U$.

Consider now $V = \left(\mathfrak{m}^K\right)^{(d)} \subseteq U_3$, for a suitable $K \in \mathbb{N}$, and $\psi := (\phi_1, \phi_2) \circ F^{-1}|_V$; we shall prove that the R-chart (V, ψ) of $\mathbf{0}$ is adapted to N. Suppose firstly that we have $x, y \in V$ such that $xy^{-1} = n \in N$. Since $x, y \in V \subseteq U_3$ there exist some unique $n_1, n_2 \in U_1$ and $h_1, h_2 \in U_2$ such that $x = n_1 h_1$ and $y = n_2 h_2$, so $n_1 h_1 = x = n n_2 h_2$. Moreover, since $n = xy^{-1} \in V \cap N$, then $n \in U_1$, that is, $nn_2 \in U_1$. Thus, since $F|_{U_1 \times U_2}$ is a bijection, we have that $h_2 = h_1$. Conversely, whenever $\mathrm{pr}_2(x) = \mathrm{pr}_2(y)$, then $x = n_1 h$ and $y = n_2 h$ for some $n_1, n_2 \in U_1$ and $h \in U_2$, so $xy^{-1} = n_1 n_2^{-1} \in N$. □

Proposition 1.59. *Let G be an R-analytic group and $N \trianglelefteq S$ a normal analytic subgroup. Then G/N has a unique R-analytic manifold structure that makes the quotient epimorphism $\pi\colon G \to G/N$ into a submersion.*

Proof. We fix some notation: $d = \dim G$, $k = \dim N$ and $\mathrm{pr}\colon \mathfrak{m}^{(d)} \to \mathfrak{m}^{(d-k)}$ is the projection onto the last $d-k$ coordinates.

We shall endow G/N with an R-analytic atlas. It suffices to give an appropriate R-chart at $1N$. By Lemma 1.58, there exists an R-chart (V, ψ) of 1 in G that is adapted to N. Therefore, since π is an open map, $\overline{V} = \pi(V)$ is open in G/N, and since the chart is adapted to N then $\mathrm{pr} \circ \psi$ is well-defined in $\overline{V}$. Consequently $\left(\overline{V}, \mathrm{pr} \circ \psi\right)$ is an R-chart of dimension $d-k$ containing $1N$.

Once we have this R-chart, using the left multiplication maps $L_{aN}\colon G/N \to G/N$ we could define the R-analytic atlas $\left\{\left(L_{aN}(\overline{V}), \mathrm{pr} \circ \psi \circ L_{a^{-1}N}\right)\right\}_{a \in G}$. These R-charts are compatible. Indeed, since the initial R-chart was adapted to N it can be seen that, if $L_{aN}\left(\overline{V}\right) \cap L_{bN}\left(\overline{V}\right) \neq \varnothing$, then for some big enough $K \in \mathbb{N}$ big enough, we have that

$$\left(\mathrm{pr} \circ \psi \circ L_{a^{-1}N}\right) \circ \left(\mathrm{pr} \circ \psi \circ L_{b^{-1}N}\right)^{-1}(x) = \mathrm{pr} \circ (\psi \circ L_{a^{-1}b} \circ \psi^{-1})(\mathbf{0}, x), \ \forall x \in \left(\mathfrak{m}^K\right)^{(d-k)}, \quad (1.12)$$

–where $L_{a^{-1}b}\colon G \to G$ is the usual left multiplication map– and so the change of coordinates map is also an R-analytic map.

Furthermore, $\pi\colon G \to G/N$ is in coordinates –with respect to the R-charts (V, ψ) and $\left(\overline{V}, \mathrm{pr} \circ \psi\right)$– simply pr, so π is a submersion at 1; and since G is an R-analytic group, then π is a submersion.

Finally, suppose that there are two R-analytic manifold structures $(G/N)_1$ and $(G/N)_2$ such that the respective quotient epimorphisms are submersions. In view of Lemma 1.38, there exist R-charts such that the maps $\pi\colon G \to (G/N)_i$ ($i \in \{1,2\}$) are given in coordinates by $(x_1, \ldots, x_d) \to (x_{k+1}, \ldots, x_d)$, and thus with respect to these R-charts $\mathrm{Id}\colon (G/N)_1 \to (G/N)_2$ is the identity in coordinates, so it is R-bianalytic and the above manifold structure is unique. $\square$

Finally, it is easy to see that with the preceding manifold structure in the quotient group the multiplication and the inversion are R-analytic maps (it is proved after the fashion of (1.12)). Thus, we have the following result:

Theorem 1.60. *Let G be an R-analytic group and let $N \trianglelefteq G$ be a normal analytic subgroup. Then, there exists a unique analytic structure in G/N making it into an R-analytic group and the quotient epimorphism $\pi\colon G \to G/N$ into a submersion.*

1.6 Notes

The definition of analytic group over a general pro-p domain appears for the first time in [24, Chapter 13], even though they were already described over complete principal ideal domains in [11] and [68]. Sections 1.1 to 1.3 include well-known generalities about those groups, and they chiefly follow [24, Chapter 13]. Subsection 1.3.1, where we describe the group structure of an analytic group using a suitable open standard subgroup, constitutes the principal original contribution.

The construction of a Lie algebra starting from a formal group law was initially done in [11], and most of the results in Section 1.4 can be regarded as the generalisation to general pro-p domains of those in [68].

The development, in Section 1.5, of the concept of submanifold and quotient manifold is new, albeit such a study is suggested in [24, pg. 349], and, in the author's opinion, it might be helpful for further investigations.

Apart from the above-specified references, numerous properties of R-analytic groups can be found in the articles [45] and [54].

Finally, as an aside, in (1.1), the definition of Λ (the distinction between PIDs and other pro-p domains) might seem somewhat arbitrary. Still, it is in line with the theory of analytic groups over local fields. For example, for the p-adic analytic group $\mathbb{Z}_p$, the charts $L_p\colon \mathbb{Z}_p \to p\mathbb{Z}_p$ and $\mathrm{Id}\colon \mathbb{Z}_p \to \mathbb{Z}_p$ ought to be compatible. When R is a *unique factorisation domain* (UFD), the distinction can be avoided by always taking $\Lambda = \mathrm{Frac}(R)$. Still, for unique factorisation pro-p domains that are not PIDs, the set of power series of $K[[\mathbf{X}]]$ that are convergent around 0 would be $R[[\mathbf{X}]]$. However, the situation needs to be clarified for arbitrary pro-p

domains. Nonetheless, [24, Section 13.6] is worth reading for a more extensive discussion on this topic.

2

Linearity of compact R-analytic groups

A *faithful linear representation* of a group G of degree $n \in \mathbb{N}$ over a field K is an injective group homomorphism $\rho\colon G \to \mathrm{GL}_n(K)$, or equivalently a faithful action of G on an n-dimensional K-vector space W.

Definition 2.1. A group G is linear when there exists a faithful linear representation of it.

Therefore, roughly speaking linear groups are subgroups of invertible matrices, and this gives rise to a rich interplay between algebra and geometry, as well as they satisfy the myriad of properties known for matrix groups. Those groups were initially introduced by Jordan [46], and the subsequent works of Jordan himself and many other prominent mathematicians pushed their systematic study ahead. Since then they have been one of the most studied families of groups, leading to many applications inside and outside group theory.

We shall make a couple of elementary observations:

Remark 2.2. (i) Linear representations can be defined alike over integral domains, but G has a faithful linear representation over a field if and only if it has a faithful linear representation over an integral domain.

(ii) Virtually linear groups are actually linear. Indeed, let $H \leq G$ be a finite index subgroup and suppose that H acts faithfully on the finite dimensional vector space W. Fix a left transversal T for H in G and let

$$V := \bigoplus_{t \in T} W_t,$$

where $W_t \cong W$. Then, V is a vector space of dimension $|G : H| \dim W$. For notational convenience we denote by (t, w) the element that corresponds to $w \in W$ inside W_t. Then, G acts on V by

$$g \cdot (t, w) = \left(\widetilde{gt}, w \cdot h\right),$$

where $\widetilde{gt}$ is the element in T representing the coclass gtH and $h = gt\widetilde{gt}^{-1} \in H$. The reader can easily verify that this gives rise to a faithful representation of G, which is referred to as the *induced representation on G*.

In the context of R-analytic groups, in his introductory paper [50] of 1965, Lazard proved that compact p-adic analytic groups are linear, and since R-analytic groups are a generalisation of p-adic analytic groups, it is a natural, as well as a long-standing question whether the same holds for general analytic groups. On the one hand, using the analogue of the adjoint representation present in the theory of Lie algebras (compare with Section A.2), we can show that every R-standard group is "linear modulo its centre".

Proposition 2.3 (cf. [42, Proposition 5.1])**.** *Let S be an R-standard group, then $S/Z(S)$ is linear over R.*

Proof. Since S is an R-standard group we can identify it with $\left(\mathfrak{m}^N\right)^{(d)}$. Let $\mathbf{F}$ and $\mathbf{I}$ be respectively the formal group law and the formal inverse of S, then the conjugation map c_g is strictly analytic, and it is given by the tuple of power series

$$\mathbf{C}_g(\mathbf{X}) = (C_1^g(\mathbf{X}), \ldots, C_d^g(\mathbf{X})) := \mathbf{F}(\mathbf{F}(\mathbf{I}(g), \mathbf{X}), g).$$

We shall present an action of S on a finitely generated R-module W. Firstly, S acts on $R[[\mathbf{X}]]$ by $g \cdot f(\mathbf{X}) := f \circ \mathbf{C}_g(\mathbf{X}) \in R[[\mathbf{X}]]$. In particular, if $\pi_i \in R[[\mathbf{X}]]$ is the ith projection power series $\pi_i(\mathbf{X}) = X_i$, by (1.5), there exist some $\pi_{i,\alpha} \in R[[\mathbf{X}]]$ such that

$$(g \cdot \pi_i)(\mathbf{X}) = C_i^g(\mathbf{X}) = X_i + \sum_{|\alpha| \geq 1} \pi_{i,\alpha}(g) X_1^{\alpha_1} \ldots X_d^{\alpha_d}.$$

Let $\mathcal{I}$ be the ideal of $R[[\mathbf{X}]]$ generated by the formal power series

$$\left\{\pi_{i,\alpha} \mid \alpha \in \mathbb{N}_0^{(d)}, i \in \{1, \ldots, d\}\right\}. \tag{2.1}$$

Since $R[[\mathbf{X}]]$ is Noetherian (see [49, Theorem 9.4]), $\mathcal{I}$ is generated by a finite subset of (2.1), say $\mathcal{F}$, and so consider $m = \max\{|\alpha| \mid \pi_{i,\alpha} \in \mathcal{F}\} \in \mathbb{N}$. Denote by $\mathfrak{M}$ the ideal $(X_1, \ldots, X_d)R[[\mathbf{X}]]$, then $W = \mathfrak{M}/\mathfrak{M}^{m+1}$ is a free R-module of finite rank, and S acts on W.

Clearly $Z(S)$ acts trivially on W. Conversely, suppose that g acts trivially on W. Then, $\pi(g) = 0$, for all $\pi \in \mathcal{F}$. Hence, $\pi_{i,\alpha}(g) = 0$ for all $i \in \{1, \ldots, d\}$ and $\alpha \in \mathbb{N}_0^{(d)}$, so $g \cdot \pi_i(\mathbf{X}) = X_i$, that is, $g \in Z(S)$. Thus, $S/Z(S)$ acts faithfully on W. □

On other hand, Lazard's proof – it will be briefly explained in Section 2.1–, relies on Ado's Theorem in conjunction with the Baker-Campbell-Hausdorff formula in order to link the group operation with the corresponding Lie algebra operation. Based on that procedure, Camina and Du Sautoy [14] proved that perfect $\mathbb{Z}_p[[t]]$-standard groups, namely $\mathbb{Z}_p[[t]]$-standard groups S of level N such that $S' = S_{2N}$ (see (1.6)), are linear. Furthermore, Jaikin-Zapirain [43] exploited similar ideas to prove that over a pro-p domain R of characteristic zero, every finitely generated compact R-analytic group is linear. The main result of this chapter is to extend this, and prove that whenever R has characteristic zero, any compact R-analytic group is linear.

2.1 Linearity of compact p-adic analytic groups

In this section, we will succinctly describe the construction of the aforementioned faithful linear representation for compact p-adic analytic groups, with a view towards controlling its degree. We start by recalling Ado's Theorem, more precisely a generalised version of it, due by Churkin [19] and Weigel [74] (see Appendix A for a detailed proof as well as for pertinent definitions on the topic).

Theorem 2.4. *Let $\mathfrak{L}$ be a $\mathbb{Z}_p$-Lie algebra which is a free $\mathbb{Z}_p$-module of rank r. There exists a $\mathbb{Z}_p$-Lie algebra monomorphism $\phi\colon \mathfrak{L} \hookrightarrow \mathrm{M}_n(\mathbb{Z}_p)$, where n depends only on r.*

Fix a prime number p, and for simplicity of notation set $\mathbf{p} = p$ if p is odd and $\mathbf{p} = 4$ if $p = 2$. Any $\mathbb{Z}_p$-standard group (S, ϕ) is a so-called *uniformly powerful group*, that is, a finitely generated torsion-free group such that $S' \leq S^{\mathbf{p}}$ (cf. [24, Thms 4.5 and 8.31]). There is a categorical isomorphism between the category $\mathfrak{UGroup}$ of d-dimensional uniformly powerful pro-p groups and the category $\mathfrak{pLie}$ of d-dimensional *powerful* $\mathbb{Z}_p$*-Lie lattices*, namely $\mathbb{Z}_p$-Lie algebras $\mathfrak{L}$ that are free $\mathbb{Z}_p$-modules of rank d such that $[\mathfrak{L}, \mathfrak{L}] \leq \mathbf{p}\mathfrak{L}$. More concretely, there exists a categorical isomorphism $\mathcal{L}\colon \mathfrak{UGroup} \to \mathfrak{pLie}$, with inverse $\mathcal{E}$, and to each uniformly powerful group S of dimension d, it assigns a $\mathbb{Z}_p$-Lie algebra $\mathcal{L}(S)$ that is a free $\mathbb{Z}_p$-module of rank d, where the underlying set is S itself, and the module operations are given in terms of the group operations as follows: let $r \in \mathbb{Z}_p$ and $x, y \in S$ then

$$r \cdot x = x^r$$
$$x + y = \lim_{n\to\infty} \left(x^{p^n} y^{p^n}\right)^{p^{-n}}$$
$$[x, y] = \lim_{n\to\infty} \left(\left[x^{p^n}, y^{p^n}\right]\right)^{p^{-2n}},$$

we refer to [24, Section 4.3] for the precise definitions of the right-hand side formulas. For instance, $\mathbf{p}\,\mathrm{M}_n(\mathbb{Z}_p)$ is a powerful Lie lattice with Lie bracket $[A, B] = AB - BA$, and $\mathcal{E}(\mathbf{p}\,\mathrm{M}_n(\mathbb{Z}_p)) \subseteq \mathrm{GL}_n(\mathbb{Z}_p)$ (this can be viewed as the usual p-adic matrix exponentiation).*

Conversely, the *Baker-Hausdorff-Campbell* formula can be regarded as a formal power series $H(x, y)$ in two non-commuting variables satisfying the identity

$$e^{H(x,y)} = e^x e^y,$$

and given a powerful $\mathbb{Z}_p$-Lie lattice $\mathcal{S}$, the set $\mathcal{S}$ is a uniformly powerful group with the group operation $xy = H(x, y)$.

Consequently, if $\phi\colon \mathfrak{L}(S) \hookrightarrow \mathrm{M}_n(\mathbb{Z}_p)$ is the injective $\mathbb{Z}_p$-Lie algebra homomorphism provided by Theorem 2.4, then

$$\phi|_{\mathbf{p}\mathcal{L}(S)}\colon \mathbf{p}\mathcal{L}(S) \to \mathbf{p}\,\mathrm{M}_n(\mathbb{Z}_p)$$

is an injective Lie algebra homomorphism between powerful $\mathbb{Z}_p$-Lie lattices, and hence, since $\mathcal{E}$ is a functor, we obtain a group monomorphism

$$\mathcal{E}(\phi)\colon \mathcal{E}(\mathbf{p}\mathcal{L}(S)) \hookrightarrow \mathcal{E}(\mathbf{p}\,\mathrm{M}_n(\mathbb{Z}_p)) \leq \mathrm{GL}_n(\mathbb{Z}_p).$$

Finally, the additive cosets of the $\mathbb{Z}_p$-Lie algebra are the same as the multiplicative cosets of the uniformly powerful group (cf. [24, Proposition 4.31(iii)]), so

$$|S : \mathcal{E}\left(\mathbf{p}\mathcal{L}(S)\right)| = |\mathcal{L}(S) : \mathcal{L} \circ \mathcal{E}(\mathbf{p}\mathcal{L}(S))| = |\mathcal{L}(S) : \mathbf{p}\mathcal{L}(S)| = \left|\mathbb{Z}_p^{(d)} : \mathbf{p}\mathbb{Z}_p^{(d)}\right| = \mathbf{p}^d,$$

*Actually the image of $\mathbf{p}\,\mathrm{M}_n(\mathbb{Z}_p)$ by $\mathcal{E}$ is the first congruence subgroup $\mathrm{GL}_n^1(\mathbb{Z}_p)$.

and thus by taking the induced representation we obtain a group monomorphism

$$m\colon S \hookrightarrow \mathrm{GL}_{\mathbf{p}^d n}(\mathbb{Z}_p).$$

We gather all this in the following result:

Theorem 2.5. *Let S be a $\mathbb{Z}_p$-standard group of dimension d. There exists a faithful linear representation $m\colon S \hookrightarrow \mathrm{GL}_n(\mathbb{Z}_p)$, whose degree n depends only on p and d.*

Finally, since every compact p-adic analytic group has a $\mathbb{Z}_p$-standard subgroup of finite index, the induced linear representation leads to:

Corollary 2.6 (Lazard)**.** *Every compact p-adic analytic group is linear over $\mathbb{Z}_p$.*

2.2 Change of pro-p domains

The idea behind various results in this thesis is to reduce the problem to analytic groups over pro-p domains of Krull dimension one and use the already known structural results there. For that purpose a homomorphism $\varphi\colon R \to Q$ between pro-p domains can be used to construct Q-analytic groups that are natural images of R-analytic groups. Indeed, given a power series $F(\mathbf{X}) = \sum_{\alpha\in\mathbb{N}_0^{(d)}} a_\alpha \mathbf{X}^\alpha \in R[[\mathbf{X}]]$ by applying φ to the coefficients we obtain the power series

$$\mathbf{F}_\varphi = \sum_{\alpha\in\mathbb{N}_0^{(d)}} \varphi(a_\alpha)\mathbf{X}^\alpha \in Q[[\mathbf{X}]].$$

This is no more than a specific instance of the wider universal property of topological power series rings (see Section 1.1).

To transform coefficients, we restrict to natural ring homomorphisms between local rings, namely *local ring homomorphisms*. Those are ring homomorphisms $\varphi\colon (R,\mathfrak{m}) \to (Q,\mathfrak{n})$ between local rings such that $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. Observe that, every local ring homomorphism is continuous, considering that $\varphi(\mathfrak{m}^n) \subseteq \mathfrak{n}^n$ for any $n \in \mathbb{N}$.

The following lemma shows that the foregoing change of rings commutes with the composition of power series.

Lemma 2.7. *Let $\varphi\colon (R,\mathfrak{m}) \to (Q,\mathfrak{n})$ be a local ring homomorphism. Let $\boldsymbol{F} \in R[[X_1,\dots,X_n]]^{(m)}$ and $\mathbf{G} \in R[[X_1,\dots,X_m]]^{(l)}$ be formal power series, and assume that $\boldsymbol{F}(\mathbf{0}) \in \mathfrak{m}^{(m)}$. Then $(\mathbf{G}\circ\boldsymbol{F})_\varphi(X_1,\dots,X_n) = \mathbf{G}_\varphi \circ \boldsymbol{F}_\varphi(X_1,\dots,X_n)$.*

Proof. Using the universal property of power series rings, there exists a unique continuous ring homomorphism

$$\Phi_\varphi \colon R[[X_1, \dots, X_n]] \to Q[[X_1, \dots, X_n]]$$

such that $\Phi_\varphi(\mathbf{H}(\mathbf{X})) = \mathbf{H}_\varphi(\mathbf{X})$ for all $\mathbf{H} \in R[[\mathbf{X}]]$, where $\mathbf{X} = (X_1, \dots, X_n)$.

Let $\mathbf{F}(\mathbf{X}) = (F_1(\mathbf{X}), \dots, F_m(\mathbf{X}))$. Since $\mathbf{F}(\mathbf{0}) \in \mathfrak{m}^{(m)}$ and φ is local, then $\Phi_\varphi(F_1(\mathbf{X})), \dots, \Phi_\varphi(F_m(\mathbf{X}))$ are in $(\mathfrak{n}, X_1, \dots, X_n)Q[[\mathbf{X}]]$, the maximal ideal of $Q[[\mathbf{X}]]$, therefore using the universal property of power series rings we can define the continuous map

$$\Phi_1 \colon R[[Y_1, \dots, Y_m]] \to Q[[X_1, \dots, X_n]]$$

such that $\Phi_1(r) = \varphi(r)$ for all $r \in R$, and $\Phi_1(Y_i) = (F_i)_\varphi(\mathbf{X})$ for all $i \in \{1, \dots, m\}$. Similarly, since $F_1(\mathbf{X}), \dots, F_m(\mathbf{X})$ are in the maximal ideal of $R[[\mathbf{X}]]$, define the map

$$\Phi_2 \colon R[[Y_1, \dots, Y_m]] \to R[[X_1, \dots, X_n]]$$

such that $\Phi_2|_R = \mathrm{Id}_R$, and $\Phi_2(Y_i) = F_i(\mathbf{X})$ for all $i \in \{1, \dots, m\}$.

Notice that $\Phi_1(r) = \Phi_\varphi \circ \Phi_2(r) = \varphi(r)$ for all $r \in R$, and that $\Phi_1(Y_i) = \Phi_\varphi \circ \Phi_2(Y_i) = (F_i)_\varphi(\mathbf{X})$ for all $i \in \{1, \dots, m\}$. Therefore, by the uniqueness of the universal property we have that $\Phi_1 = \Phi_\varphi \circ \Phi_2$, and so, in particular,

$$(\mathbf{G} \circ \mathbf{F})_\varphi(\mathbf{X}) = \Phi_\varphi \circ \Phi_2(\mathbf{G}(\mathbf{Y})) = \Phi_1(\mathbf{G}(\mathbf{Y})) = \mathbf{G}_\varphi \circ \mathbf{F}_\varphi(\mathbf{X}). \qquad \square$$

Hence, the change of rings preserves formal power series identities, so in particular:

Corollary 2.8. *Let R and Q be pro-p domains, let $\boldsymbol{F} \in R[[X_1, \dots, X_{2d}]]^{(d)}$ be a formal group law with formal inverse $\boldsymbol{I}$ and let $\varphi \colon R \to Q$ be a local ring homomorphism. Then $\boldsymbol{F}_\varphi$ is a formal group law with formal inverse $\boldsymbol{I}_\varphi$.*

Proof. Let $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ be d-tuples of variables. Since $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and using Lemma 2.7 we have:

(i) since $\mathbf{F}(\mathbf{F}(\mathbf{X}, \mathbf{Y}), \mathbf{Z}) = \mathbf{F}(\mathbf{X}, \mathbf{F}(\mathbf{Y}, \mathbf{Z}))$, then

$$\mathbf{F}_\varphi(\mathbf{F}_\varphi(\mathbf{X}, \mathbf{Y}), \mathbf{Z}) = \mathbf{F}_\varphi(\mathbf{X}, \mathbf{F}_\varphi(\mathbf{Y}, \mathbf{Z})),$$

(ii) and since $\mathbf{F}(\mathbf{X}, \mathbf{0}) = \mathbf{F}(\mathbf{0}, \mathbf{X}) = \mathbf{X}$, then $\mathbf{F}_\varphi(\mathbf{X}, \mathbf{0}) = \mathbf{F}_\varphi(\mathbf{0}, \mathbf{X}) = \mathbf{X}$.

Therefore $\mathbf{F}_\varphi \in Q[[X_1,\dots,X_{2d}]]^{(d)}$ is a formal group law. Finally, since $\mathbf{I}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{F}(\mathbf{I}(\mathbf{X}),\mathbf{X}) = \mathbf{F}(\mathbf{X},\mathbf{I}(\mathbf{X})) = \mathbf{0}$, by Lemma 2.7:

$$\mathbf{F}_\varphi\left(\mathbf{I}_\varphi(\mathbf{X}),\mathbf{X}\right) = \mathbf{F}_\varphi\left(\mathbf{X},\mathbf{I}_\varphi(\mathbf{X})\right) = \mathbf{0},$$

and so $\mathbf{I}_\varphi$ is the formal inverse corresponding to $\mathbf{F}_\varphi$. □

Let now $\varphi\colon (R,\mathfrak{m}) \to (Q,\mathfrak{n})$ be a local ring homomorphism between pro-p domains and let S be an R-standard group, which can be identified for simplicity with $\left(\mathfrak{m}^N\right)^{(d)}$, whose formal group law is $\mathbf{F}$. Using the above-constructed formal group law $\mathbf{F}_\varphi$, $L := \left(\mathfrak{n}^N\right)^{(d)}$ can be endowed with a group operation making it into a Q-standard group. Indeed, the group operation is simply

$$x * y = \mathbf{F}_\varphi(x,y), \tag{2.2}$$

for all $x, y \in L$. In the following lemma we keep all the preceding notation.

Lemma 2.9. *The map*

$$\varphi^{(d)}\colon \left(\left(\mathfrak{m}^N\right)^{(d)}, \boldsymbol{F}\right) \to \left(\left(\mathfrak{n}^N\right)^{(d)}, \boldsymbol{F}_\varphi\right),\ (r_1,\dots,r_d) \mapsto (\varphi(r_1),\dots,\varphi(r_d))$$

is a group homomorphism.

Proof. If we write $\mathbf{F} = (F_1,\dots,F_d)$ and

$$F_i(\mathbf{X},\mathbf{Y}) = \sum_{\alpha,\beta\in\mathbb{N}_0^{(d)}} a_{\alpha,\beta}\mathbf{X}^\alpha\mathbf{Y}^\beta \in R[[\mathbf{X},\mathbf{Y}]],$$

then

$$\begin{aligned}
\varphi(F_i(x,y)) &= \varphi\left(\sum_{\alpha,\beta\in\mathbb{N}_0^{(d)}} a_{\alpha,\beta}x_1^{\alpha_1}\dots x_d^{\alpha_d}y_1^{\beta_1}\dots y_d^{\beta_d}\right)\\
&= \sum_{\alpha,\beta\in\mathbb{N}_0^{(d)}} \varphi(a_{\alpha,\beta})\varphi(x_1)^{\alpha_1}\dots\varphi(x_d)^{\alpha_d}\varphi(y_1)^{\beta_1}\dots\varphi(y_d)^{\beta_d}\\
&= (F_i)_\varphi\left(\varphi^{(d)}(x),\varphi^{(d)}(y)\right)\ \forall x,y\in\left(\mathfrak{m}^N\right)^{(d)},
\end{aligned}$$

using the continuity of φ in the second equality; and consequently $\varphi^{(d)}$ is a group homomorphism. □

Suppose now that R is not a PID. Let G be a compact R-analytic group and let $(S, \mathbf{F})$ be an open normal R-standard group such that the conjugation maps are strictly analytic (we can assure its existence by virtue of Lemma 1.23). We shall construct a compact Q-analytic group, whose open normal Q-standard subgroup will be the Q-standard group $L = \left(\left(\mathfrak{n}^N\right)^{(d)}, \mathbf{F}_\varphi\right)$ built upon S as in (2.2). More precisely, let T be a left transversal for S in G, and assume that $1 \in T$. For notational convenience, we will use the following: whenever $g \in G$ then $\tilde{g}$ is the representative of gS in T. By (1.3.1), and using the notation therein, if $x \in tS$ and $y \in rS$, their product is given by

$$\phi_{\widetilde{tr}}(xy) = A_{tr}^{\widetilde{tr}}\left(\mathbf{F}\left(\mathbf{C}_r(\phi_t(x)), \phi_r(y)\right)\right).$$

Define $H := T \times L$ and the homeomorphisms $\psi_t\colon (t, L) \to L$, $\psi_t(t, l) \mapsto l$. If $x \in (t, L)$ and $y \in (r, L)$, imitating the previous formula define the operation:

$$x * y = \left(\widetilde{tr}, \left(A_{tr}^{\widetilde{tr}}\right)_\varphi \left(\mathbf{F}_\varphi\left(\left(\mathbf{C}_r\right)_\varphi(\psi_t(x)), \psi_r(y)\right)\right)\right). \tag{2.3}$$

Remark. We can identify $(1, L)$ with L for plainness, and then (2.3) extends (2.2).

Lemma 2.10. *With the notation above, $(H, *)$ is a compact Q-analytic group with open normal Q-standard subgroup L.*

Proof. Firstly, H is a compact Q-analytic manifold with atlas $\{(tL, \psi_t)\}_{t \in T}$ – abusing the notation we will use tL to denote (t, L)–. Moreover, H is a group. Indeed,

(i) let $t, r, p \in T$, from Lemma 1.24 and the associativity of G we know that as formal power series:

$$A_{t\cdot\widetilde{rp}}^{\widetilde{trp}}\left(\mathbf{F}\left(\mathbf{C}_{\widetilde{rp}}(\mathbf{X}), A_{rp}^{\widetilde{rp}}(\mathbf{F}(\mathbf{C}_p(\mathbf{Y}), \mathbf{Z}))\right)\right) = A_{\widetilde{tr}\cdot p}^{\widetilde{trp}}\left(\mathbf{F}\left(\mathbf{C}_p\left(A_{tr}^{\widetilde{tr}}\left(\mathbf{F}\left(\mathbf{C}_r(\mathbf{X}), \mathbf{Y}\right)\right)\right), \mathbf{Z}\right)\right).$$

Thus, let $x \in tL$, $y \in rL$ and $z \in pL$, then by Lemma 2.7 and (2.3):

$$\begin{aligned}
&\psi_{\widetilde{trp}}(x * (y * z)) \\
&= \left(A_{t\cdot\widetilde{rp}}^{\widetilde{trp}}\right)_\varphi \left(\mathbf{F}_\varphi\left(\left(\mathbf{C}_{\widetilde{rp}}\right)_\varphi(\psi_t(x)), \left(A_{rp}^{\widetilde{rp}}\right)_\varphi\left(\mathbf{F}_\varphi\left(\left(\mathbf{C}_p\right)_\varphi(\psi_r(y)), \psi_p(z)\right)\right)\right)\right) \\
&= \left(A_{\widetilde{tr}\cdot p}^{\widetilde{trp}}\right)_\varphi \left(\mathbf{F}_\varphi\left(\left(\mathbf{C}_p\right)_\varphi\left(\left(A_{tr}^{\widetilde{tr}}\right)_\varphi\left(\mathbf{F}_\varphi\left(\left(\mathbf{C}_r\right)_\varphi(\psi_t(x)), \psi_r(y)\right)\right)\right), \psi_p(z)\right)\right) \\
&= \psi_{\widetilde{trp}}((x * y) * z).
\end{aligned}$$

(ii) The neutral element is $(1, \mathbf{0}) \in L$.

(iii) The inverse of $x \in tL$ is given by

$$y = \left(\widetilde{t^{-1}}, \left(A_{t^{-1}}^{\widetilde{t^{-1}}}\right)_{\varphi} \circ (\mathbf{C}_{t^{-1}})_{\varphi} \circ \mathbf{I}_{\varphi}(\psi_t(x))\right).$$

Indeed, clearly $x * y \in L$ and by Lemma 1.24, we know that

$$A^1_{t \cdot \widetilde{t^{-1}}} \left(\mathbf{F}\left(\mathbf{C}_{\widetilde{t^{-1}}}(\mathbf{X}), A_{t^{-1}}^{\widetilde{t^{-1}}}\left(\mathbf{C}_{t^{-1}}(\mathbf{I}(\mathbf{X}))\right)\right)\right) = \mathbf{0}.$$

Hence, by Lemma 2.7 and (2.3),

$$\begin{aligned}
\mathbf{0} &= \left(A^1_{t \cdot \widetilde{t^{-1}}}\right)_{\varphi} \left(\mathbf{F}_{\varphi}\left(\left(\mathbf{C}_{\widetilde{t^{-1}}}\right)_{\varphi}(\psi_t(x)), \left(A_{t^{-1}}^{\widetilde{t^{-1}}}\right)_{\varphi}\left((\mathbf{C}_{t^{-1}})_{\varphi}\left(\mathbf{I}_{\varphi}(\psi_t(x))\right)\right)\right)\right) \\
&= \left(A^1_{t \cdot \widetilde{t^{-1}}}\right)_{\varphi} \left(\mathbf{F}_{\varphi}\left(\left(\mathbf{C}_{\widetilde{t^{-1}}}\right)_{\varphi}(\psi_t(x)), \psi_{\widetilde{t^{-1}}}(y)\right)\right) \\
&= \psi_1(x * y).
\end{aligned}$$

In a similar fashion, $y * x = (1, \mathbf{0})$.

Finally, L is normal in H, as $1^t = 1$ for all $t \in T$. □

We will finish with a trivial (but helpful in upcoming chapters) observation:

Remark 2.11. By definition, if we have that

$$\phi_{\widetilde{tr}}(x \cdot y) = \mathbf{M}\left(\phi_t(x), \phi_r(y)\right) \quad \forall (x, y) \in tS \times rS.$$

for a suitable tuple of power series $\mathbf{M} \in R[[X_1, \dots, X_{2d}]]^{(d)}$. Then,

$$\psi_{\widetilde{tr}}(\bar{x} * \bar{y}) = \mathbf{M}_{\varphi}\left(\psi_t(\bar{x}), \psi_r(\bar{y})\right) \quad \forall (\bar{x}, \bar{y}) \in tL \times rL.$$

Moreover, the same holds for the inversion map.

2.2.1 Evaluation epimorphisms

In practise, for the aforesaid change of pro-p domains we will chiefly use *evaluation epimorphisms*. More precisely, let $(R, \mathfrak{m})$ be a pro-p domain and $a \in \mathfrak{m}^{(m)}$, the evaluation epimorphism at a is the continuous local ring homomorphism $s_a\colon R[[t_1, \dots, t_m]] \to R$, $F \mapsto F(a)$. Those epimorphisms can be extended to any integral extension of $R[[t_1, \dots, t_m]]$, by virtue of the following classical result:

Theorem 2.12 (Going Up Theorem (cf. [75, Theorem V.2.3])). *Let $A \subseteq B$ be an integral ring extension. For every prime ideal $\mathfrak{p} \subseteq A$ there exists a prime ideal $\mathfrak{q} \subseteq B$ such that $\mathfrak{q} \cap A = \mathfrak{p}$.*

Corollary 2.13. *Let $A \subseteq B$ be an integral ring extension, let P be an integral domain and let $\varphi\colon A \to P$ be a ring epimorphism. There exists an integral domain Q such that φ extends to a ring epimorphism $\tilde{\varphi}\colon B \to Q$.*

Proof. Let $\mathfrak{p} = \ker \varphi$, by the Going Up Theorem, there exists a prime ideal $\mathfrak{q} \subseteq B$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. Thus, the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{\tilde{\varphi}} & B/\mathfrak{q} \\ \Big\uparrow & & \Big\uparrow{\scriptstyle \psi} \\ A & \xrightarrow{\varphi} & A/\mathfrak{p} \end{array}$$

where $\psi(x+\mathfrak{p}) = x+\mathfrak{q}$ is injective. Identifying $A/\mathfrak{p}$ with P, then $\tilde{\varphi}$ extends φ. □

Remark 2.14. The proof above gives more information about Q. Actually, since B is an integral extension of A, Q is also an integral extension of P. Indeed, any A-integral dependence in B, remains so modulo $\mathfrak{q}$: it is now an $A/\mathfrak{p}$-integral dependence in $B/\mathfrak{q}$.

Furthermore, if B is a pro-p domain, so is Q as a quotient of B by a prime ideal.

Finally, if $A \subseteq B$ is besides a finitely generated ring extension, then $B/\mathfrak{q}$ is a finitely generated $A/\mathfrak{p}$-module. Indeed, reducing the generators of B as A-module modulo $\mathfrak{q}$, we obtain a generating set for $B/\mathfrak{q}$ as $A/\mathfrak{p}$-module.

These extended evaluation epimorphisms appear in any pro-p domain R. Recall that by virtue of Cohen's Structure Theorem, R is a finitely generated, and so integral, extension of $P[[t_1,\dots,t_m]]$ where $m = \dim_{\mathrm{Krull}}(R) - 1$ and $(P,\mathfrak{n})$ is a pro-p domain of Krull dimension one – actually we can specify even more by recollecting that P is either $\mathbb{Z}_p$ or $\mathbb{F}_p[[t]]$ depending on the characteristic of R–, and therefore for each $a \in \mathfrak{n}^{(m)}$ we obtain a continuous epimorphism $\tilde{s}_a\colon R \to Q_a$. Lastly, we shall present a property that resembles Lemma 1.8. In keeping with prior notation:

Corollary 2.15. *Let $U \subseteq_o \mathfrak{n}^{(m)}$ and D a dense subset of U, then $\bigcap_{a\in D} \ker \tilde{s}_a = \{0\}$.*

Proof. We write $A = P[[t_1,\dots,t_m]]$, $\mathfrak{p}_a = \ker s_a$ and $\mathfrak{q}_a = \ker \tilde{s}_a$ for all $a \in \mathfrak{n}^{(m)}$. First of all, evaluating a power series $F \in P[[t_1,\dots,t_m]]$ is continuous, so if

$F(a) = 0$ for all $a \in D$, then $F(a) = 0$ for all $a \in U$. Moreover, by construction, $\mathfrak{p}_a = \mathfrak{q}_a \cap A$, so Lemma 1.8 yields that

$$(\cap_{a\in D}\ \mathfrak{q}_a) \cap A = \cap_{a\in D}\ \mathfrak{p}_a = \cap_{a\in U}\ \mathfrak{p}_a = \{0\}.$$

Suppose by contradiction that there exists $r \in \cap_{a\in D}\ \mathfrak{q}_a \setminus \{0\}$. Since the ring extension $A \subseteq R$ is integral, there exists a monic polynomial $f_r(X) \in A[X]$, such that

$$f_r(r) = r^n + \sum_{i=0}^{n-1} a_i r^i = 0.$$

We can additionally assume f_r to be of minimal degree. But since $\cap_{a\in D}\ \mathfrak{q}_a$ is an ideal of R, we have that $a_0 \in \cap_{a\in D}\ \mathfrak{q}_a \cap A = \{0\}$, which contradicts the minimality of n, since we could consider $f_r(X) = X^{n-1} + \sum_{i=1}^{n-1} a_i X^{i-1}$. □

2.3 Discrimination

The purpose of this section is to study the following concept:

Definition 2.16. Let A and B be two instances of the same algebraic structure, A is said to be *fully residually* B or A is *discriminated* by B – equivalently, B discriminates A– if for any finite subset $S \subseteq A$, there exists a homomorphism $h\colon A \to B$ in the corresponding category such that the restriction $h|_S$ is injective.

More generally, a family $\mathcal{B} = \{B_i\}_{i\in I}$ discriminates A or A is fully residually-$\mathcal{B}$, if for each finite subset $S \subseteq A$ there exists a morphism $h\colon A \to B_i$, for some $i \in I$, such that $h|_S$ is injective.

Example 2.17. Let us exemplify this with rings: let A and $\mathcal{B} = \{B_i\}_{i\in I}$ be rings and suppose that A is an integral domain. Then A being fully residually-$\mathcal{B}$ is equivalent to the existence of a collection of homomorphisms $\mathcal{F} \subseteq \cup_{i\in I} \operatorname{Hom}(A, B_i)$ such that

$$\bigcap_{f\in\mathcal{F}} \ker f = \{0\}.$$

The necessity is clear, since otherwise there would be a non-zero element $x \in A$ such that $f(x) = 0$ for any ring homomorphism $f\colon A \to B_i$, and so no homomorphism would be injective when restricted to $\{0, x\}$.

For the sufficiency, let $S = \{r_i\}_{i\in I} \subseteq A$ be a finite set, and define $r = \prod_{i\neq j}(r_i - r_j) \in R$. Notice that $r \neq 0$, as the r_i's are distinct and A is an integral domain.

Therefore, there exists $f \in \mathrm{Hom}(A, B_i)$ such that

$$0 \neq f(r) = \prod_{i \neq j} \left(f(r_i) - f(r_j) \right),$$

and thus $f(r_i) \neq f(r_j)$ for all $i \neq j \in I$.

For instance, Lemma 1.8 yields that the pro-p domain R discriminates $R[[\mathbf{X}]]$ for any finite number of variables in $\mathbf{X}$.

Lemma 2.18. *Let R be a pro-p domain. For each finite $S \subseteq R$ there exists a pro-p domain Q of Krull dimension one and characteristic* $\operatorname{char} R$, *and a local ring epimorphism $\varphi\colon R \to Q$ that is injective when restricted to S. In particular, a pro-p domain R is discriminated by the set of pro-p domains of Krull dimension one and characteristic* $\operatorname{char} R$.

Proof. Let $m = \dim_{\mathrm{Krull}}(R) - 1$ and let $(P, \mathfrak{n})$ be the pro-p domain $\mathbb{Z}_p$ if $\operatorname{char}(R) = 0$ or $\mathbb{F}_p[[t]]$ if $\operatorname{char}(R) = p$ is positive. According to Subsection 2.2.1, for each $a \in \mathfrak{n}^{(m)}$, there exists a pro-p domain Q_a and a local ring epimorphism $\tilde{s}_a\colon R \to Q_a$ that extends the evaluation homomorphism $s_a\colon P[[t_1, \dots, t_m]] \to P$. Moreover, by Remark 2.14, Q_a is an integral extension of P, and thus

$$\dim_{\mathrm{Krull}} Q_a = \dim_{\mathrm{Krull}} P = 1$$

and

$$\operatorname{char} Q_a = \operatorname{char} P = \operatorname{char} R.$$

Finally, by Corollary 2.15, $\bigcap_{a \in \mathfrak{n}^{(m)}} \ker \tilde{s}_a = \{0\}$, and Example 2.17 yields the result. □

As mentioned in the preceding proof, if R has characteristic zero, it is a finitely generated extension of $\mathbb{Z}_p[[t_1, \dots, t_m]]$. Let us denote by $\mu(R)$ the minimum number of elements that is necessary to generate R as a $\mathbb{Z}_p[[t_1, \dots, t_m]]$-module. We recall from Remark 2.14 that Q_a is a free $\mathbb{Z}_p$-module of rank at most $\mu(R)$.

Proposition 2.19. *Let $(R, \mathfrak{m})$ be a pro-p domain of characteristic zero, and let G be an R-standard group. There exists an integer $n \in \mathbb{N}$, depending on R, the dimension of G and the level of G, such that G is discriminated by* $\mathrm{GL}_n(\mathbb{Z}_p)$.

Proof. We identify G with $\left(\mathfrak{m}^N\right)^{(d)}$, where the group operation is given by the formal group law

$$\mathbf{F}(\mathbf{X}, \mathbf{Y}) = \sum_{\alpha, \beta \in \mathbb{N}_0^{(d)}} a_{\alpha,\beta} \mathbf{X}^\alpha \mathbf{Y}^\beta \in R[[\mathbf{X}, \mathbf{Y}]].$$

Let $S \subseteq \left(\mathfrak{m}^N\right)^{(d)}$ be a finite set of d-tuples, i.e.

$$S = \left\{r_i = (r_{i1}, \ldots, r_{id}) \in R^{(d)}\right\}_{i \in I},$$

and set $S' = \{r_{ij} \mid i \in I,\ j \in \{1, \ldots, d\}\} \subseteq R$. By Lemma 2.18, there exist a pro-p domain Q_S of Krull dimension one and characteristic zero and a local ring homomorphism $\pi_S \colon (R, \mathfrak{m}) \to (Q_S, \mathfrak{n})$, which is injective when restricted to S'.

Let H_S be the group $\left(\mathfrak{n}^N\right)^{(d)}$, which is so with the formal group law

$$\mathbf{F}_{\pi_S}(\mathbf{X}) = \sum_{\alpha, \beta \in \mathbb{N}_0^{(d)}} \pi_S(a_{\alpha,\beta}) \mathbf{X}^\alpha \mathbf{Y}^\beta \in Q[[\mathbf{X}, \mathbf{Y}]],$$

(compare with Corollary 2.8) and the natural Q_S-analytic manifold structure. According to Lemma 2.9,

$$\pi_S^{(d)} \colon G \to H_S,\ (r_1, \ldots, r_d) \mapsto (\pi_S(r_1), \ldots, \pi_S(r_d))$$

is a group homomorphism that is clearly injective when restricted to S.

Moreover, Q_S is a free $\mathbb{Z}_p$-module of rank $\mu'_S \leq \mu(R)$, and thus, by restriction of scalars (compare with Corollary 1.44), H_S is a p-adic analytic group of dimension $\mu'_S d \leq \mu(R) d$. More precisely, if $\sigma \colon Q_S \to \mathbb{Z}_p^{(\mu'_S)}$ is a $\mathbb{Z}_p$-module isomorphism, then

$$\left(p^N \mathbb{Z}_p\right)^{(\mu'_S)} = \sigma\left(p^N Q_S\right) \subseteq \sigma\left(\mathfrak{n}^N\right).$$

Thus, H_S contains the open $\mathbb{Z}_p$-standard subgroup

$$U_S := \left(\sigma^{(d)}\right)^{-1}\left(p^N \mathbb{Z}_p^{(\mu'_S d)}\right).$$

Besides, by Lemma 1.20, the group index coincides with the index as ideals (i.e. as additive subgroups), and therefore:

$$|H_S : U_S| \leq \left|\mathbb{Z}_p^{(\mu'_S d)} : \left(p^N \mathbb{Z}_p\right)^{(\mu'_S d)}\right| = p^{N \mu'_S d} \leq p^{N \mu(R) d}.$$

Lastly, by Theorem 2.5, there exists a faithful linear representation $m_1 \colon U_S \hookrightarrow \mathrm{GL}_{\gamma(\mu(R)d)}(\mathbb{Z}_p)$, for some function $\gamma \colon \mathbb{N} \to \mathbb{N}$. Let $n = p^{N\mu(R)d} \gamma(\mu(R)d)$ and let $m \colon H_S \hookrightarrow \mathrm{GL}_n(\mathbb{Z}_p)$ be the induced faithful linear representation. Then, the composition $m \circ \pi_S^{(d)} \colon G \to \mathrm{GL}_n(\mathbb{Z}_p)$ is a group homomorphism that is injective when restricted to S. □

In other words,

Corollary 2.20. *Every R-standard group is discriminated by a linear group.*

The main result in this section, as well as in the whole chapter, is the upcoming Theorem 2.27. Let us explore first a heuristic approach that we will make rigorous during this section. Based on the previous results, for any R-standard group G we obtain an embedding

$$G \hookrightarrow \prod_{\substack{S \subseteq G \\ S \text{ finite}}} \mathrm{GL}_n(\mathbb{Z}_p) \cong \mathrm{GL}_n\left(\prod_{\substack{S \subseteq G \\ S \text{ finite}}} \mathbb{Z}_p\right).$$

Yet, the infinite Cartesian power of $\mathbb{Z}_p$ is not an integral domain, so we shall modify the embedding. Our proof has model theoretical flavour, and we refer to [17] for general background and basic definitions on the topic. In spite of the fact that nearly all the results might be proved just by giving adequate references from the previous book, we will account for basic definitions and elementary properties.

Definition 2.21. Given an infinite set I a *filter* $\mathcal{U}$ in I is a family $\mathcal{U} \subseteq \mathcal{P}(I)$ such that

(U1) $\varnothing \notin \mathcal{U}$.

(U2) if $A \in \mathcal{U}$ and $A \subseteq B$, then $B \in \mathcal{U}$

(U3) if $A, B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$.

A filter $\mathcal{U}$ is an *ultrafilter* when it also satisfies that

(U4) whenever $A \subseteq I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

Finally, an ultrafilter is said to be *non-principal* when it also satisfies that

(U5) $\{i\} \notin \mathcal{U}$ for all $i \in I$.

Remark 2.22. We shall state a couple of basic properties of filters:

(i) If $\mathcal{U}$ is a filter in I then only one of A and $I \setminus A$ can be in $\mathcal{U}$. Hence, if $\mathcal{U}$ is an ultrafilter for each $A \subseteq I$ one, and only one, of A or $I \setminus A$ is in $\mathcal{U}$.

(ii) If $\mathcal{U}$ is an ultrafilter on I and $A \cup B \in \mathcal{U}$, then either A or B is in $\mathcal{U}$. Otherwise, by (U4), $I \setminus A$ and $I \setminus B$ would be in $\mathcal{U}$, and so, by (U3) and de Morgan law:

$$I \setminus (A \cup B) = (I \setminus A) \cap (I \setminus B) \in \mathcal{U},$$

contradicting (i).

According to the Ultrafilter Theorem (see [17, Corollary 1.4.4]), every family $\mathcal{F} \subseteq \mathcal{P}(I)$ satisfying the *finite intersection property*, namely

$$F_1 \cap \cdots \cap F_n \neq \varnothing \ \forall n \in \mathbb{N}, \ \forall F_1, \ldots, F_n \in \mathcal{F},$$

can be extended to an ultrafilter on I. Therefore, given an infinite set G, it is easy to come by an ultrafilter in

$$\mathcal{P}_{\text{fin}}(G) := \{S \subseteq G \mid S \text{ is finite}\} \subseteq \mathcal{P}(G).$$

Indeed, for each $S \in \mathcal{P}_{\text{fin}}(G)$ define $A_S = \{T \in \mathcal{P}_{\text{fin}}(G) \mid S \subseteq T\}$, and notice that for all $S, T \in \mathcal{P}_{\text{fin}}(G)$

$$A_S \cap A_T = A_{T \cup S}.$$

Thus the family $\mathcal{V} = \{A_S \mid S \in \mathcal{P}_{\text{fin}}(G)\}$ satisfies the finite intersection property, and by the Ultrafilter Theorem there exists an ultrafilter $\mathcal{U}$ extending $\mathcal{V}$, which, when required, we will refer to as the *natural ultrafilter in* $\mathcal{P}_{\text{fin}}(G)$. This ultrafilter is non-principal. Indeed, for any finite subset $S \subsetneq G$ take $g \in G \setminus S$, then

$$A_{\{g\}} \subseteq \mathcal{P}_{\text{fin}}(G) \setminus \{S\},$$

so, by (U2), $\mathcal{P}_{\text{fin}}(G) \setminus \{S\} \in \mathcal{U}$, and thus, by Remark 2.22(i), we have $\{S\} \notin \mathcal{U}$.

Definition 2.23. Let I be an infinite set, a family $\mathcal{G} = \{G_i\}_{i \in I}$ and $\mathcal{U}$ an ultrafilter in I. The *ultraproduct* of $\mathcal{G}$ is the Cartesian product $\prod_{i \in I} G_i$ modulo the equivalence relation

$$(g_i)_{i \in I} \sim (h_i)_{i \in I} \iff \{i \in I \mid g_i = h_i\} \in \mathcal{U}.$$

This structure will be denoted by $\prod_{i \in I} G_i / \mathcal{U}$, and the equivalence class of the tuple $(g_i \mid i \in I)$ will be denoted by $[g_i \mid i \in I]_{\mathcal{U}}$.

Comparably, an *ultrapower* of G is an ultraproduct where $G_i = G$ for all $i \in I$, which henceforward will be denoted simply by $G^{\mathcal{U}} = \prod_{i \in I} G / \mathcal{U}$.

The ultraproduct of a family of groups (resp. any other algebraic structure) is itself a group (resp. the corresponding algebraic structure), and it is so with the componentwise multiplication modulo the preceding equivalence relation. This can be proved directly by using elementary definitions, though it also follows directly by Łoś' Theorem.

Lemma 2.24. *Let $\mathcal{U}$ be an ultrafilter in I.*

(i) *If* $\{G_i\}_{i\in I}$ *is a family of groups, then* $\bar{G} := \prod_{i\in I} G_i/_{\mathcal{U}}$ *is a group.*

(ii) *If* $\{R_i\}_{i\in I}$ *is a family of rings, then* $\bar{R} := \prod_{i\in I} R_i/_{\mathcal{U}}$ *is a ring. Moreover, if all the* R_i*'s are integral domains, so is* $\overline{R}$*.*

Proof. (i) Note that $\bar{G}$ is the quotient of $\prod_{i\in I} G_i$ by the normal subgroup

$$\mathcal{N} = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} G_i \middle| \{i \in I \mid x_i = 1\} \in \mathcal{U} \right\}.$$

We shall verify that $\mathcal{N}$ is a normal subgroup. For each $x = (x_i \mid i \in I)$ we abbreviate by J_x the set $\{i \in I \mid x_i = 1\} \subseteq I$. Take $x = (x_i \mid i \in I)$ and $y = (y_i \mid i \in I) \in \mathcal{N}$, that is, $J_x, J_y \in \mathcal{U}$. Moreover,

$$J_x \cap J_y = J_x \cap J_{y^{-1}} \subseteq J_{xy^{-1}},$$

so, in view of (U2) and (U3), $J_{xy^{-1}} \in \mathcal{U}$ and so $xy^{-1} \in \mathcal{N}$. Similarly, for every $g \in \prod_{i\in I} G_i$ we have that $J_x \subseteq J_{x^g}$, and therefore $\mathcal{N}$ is a normal subgroup.

(ii) That $\bar{R}$ is a ring is proved like in (i), that is, $\bar{R}$ is the quotient of $\prod_{i\in I} R_i$ by the ideal

$$\mathcal{I} = \left\{ r \in \prod_{i\in I} R_i \middle| \{i \in I \mid r_i = 0\} \in \mathcal{U} \right\}.$$

For the second part, we shall prove that $\mathcal{I}$ is a prime ideal provided that each R_i is an integral domain. Obviously, $\bar{R} \neq \{0\}$, and let $r = [r_i \mid i \in I]_{\mathcal{U}}, s = [s_i \mid i \in I]_{\mathcal{U}} \in \bar{R}$ such that

$$[r_i s_i \mid i \in I]_{\mathcal{U}} = rs \in \mathcal{I},$$

that is, $J := \{i \in I \mid r_i s_i = 0\} \in \mathcal{U}$. Define $J_r = \{i \in I \mid r_i = 0\}$ and $J_s = \{i \in I \mid s_i = 0\}$. Since each R_i is an integral domain, $J = J_r \cup J_s$, so, by Remark 2.22(ii), either $J_r \in \mathcal{U}$ or $J_s \in \mathcal{U}$, that is, either $r \in \mathcal{I}$ or $s \in \mathcal{I}$. □

Now we can modify the initial heuristic argument:

Proposition 2.25. *Every R-standard group can be embedded in an ultrapower of* $\mathrm{GL}_n(\mathbb{Z}_p)$ *for a convenient* n*.*

Proof. According to Proposition 2.19, there exists $n \in \mathbb{N}$ such that for each finite subset $S \subseteq G$, there exists a group homomorphism $h_S \colon G \to \mathrm{GL}_n(\mathbb{Z}_p)$ that is injective when restricted to S.

Let $\mathcal{U}$ be the natural ultrafilter in $\mathcal{P}_{\mathrm{fin}}(G)$, and define the group homomorphism

$$h \colon G \to \mathrm{GL}_n(\mathbb{Z}_p)^{\mathcal{U}},\ g \mapsto [h_S(g) \mid S \in \mathcal{P}_{\mathrm{fin}}(G)]_{\mathcal{U}}\,.$$

We have to prove that $\ker h = \{1\}$. Let $g \neq 1$. Then $h_S(g) \neq 1$ for all $S \in A_{\{1,g\}}$, so

$$Y := \{S \in \mathcal{P}_{\mathrm{fin}}(G) \mid h_S(g) = 1\} \subseteq \mathcal{P}_{\mathrm{fin}}(G) \setminus A_{\{1,g\}},$$

and Remark 2.22(i) forces $Y \notin \mathcal{U}$, that is, $h(g) \neq 1$. Hence, h is an embedding into $\mathrm{GL}_n(\mathbb{Z}_p)^{\mathcal{U}}$. □

We will also need the following observation:

Lemma 2.26. *Let $\{R_i\}_{i\in I}$ be a family of rings and $\mathcal{U}$ an ultrafilter in I. Then,*

$$\prod_{i\in I} \mathrm{GL}_n(R_i)/\mathcal{U} \cong \mathrm{GL}_n\left(\prod_{i\in I} R_i/\mathcal{U}\right),$$

as groups.

Proof. It is an uncomplicated exercise checking that

$$\begin{aligned} f \colon \prod_{i\in I} \mathrm{GL}_n(R_i)/\mathcal{U} &\longrightarrow \mathrm{GL}_n\left(\prod_{i\in I} R_i/\mathcal{U}\right) \\ \left[A^i = (a^i_{j,k})_{j,k} \mid i \in I\right]_{\mathcal{U}} &\longmapsto \left(\left[a^i_{jk} \mid i \in I\right]_{\mathcal{U}}\right)_{j,k} \end{aligned}$$

is a group isomorphism. □

Finally, we can gather all the ingredients.

Theorem 2.27. *Let R be a pro-p domain of characteristic zero. Every compact R-analytic group is linear.*

Proof. Since every compact R-analytic group contains an R-standard group of finite index (compare with Lemma 1.21), by considering the induced linear representation, without loss of generality assume that G is R-standard. According to Theorem 2.25, G embeds in an ultrapower $\mathrm{GL}_n(\mathbb{Z}_p)^{\mathcal{U}}$ for a convenient $n \in \mathbb{N}$. Moreover, Lemma 2.26 leads to

$$G \hookrightarrow \mathrm{GL}_n(\mathbb{Z}_p)^{\mathcal{U}} \cong \mathrm{GL}_n\left(\mathbb{Z}_p^{\mathcal{U}}\right),$$

and by Lemma 2.24(ii) $\mathbb{Z}_p^{\mathcal{U}}$ is an integral domain, so G is linear. □

2.5 Notes

The strategies behind the proofs of Section 2.1 are due to Lazard, although the exposition here is quite different, and follows [24, Section 7.3]; nevertheless the description is more detailed in the book. The vast majority of the results in Section 2.4 are basic for any model theorist, but we have included them for the sake of completeness. As it is shown here, one of the applications of model theory outside its realm is proving that certain groups are linear, which in general is not an easy task. This usage was firstly observed by Mal'cev [55].

The rest of the material is original, it has been obtained in collaboration with Casals-Ruiz and is published in [16], albeit some parts appear in [76].

3

Hausdorff dimension in compact R-analytic groups

FRACTAL DIMENSIONS arose as a generalisation of the notion of topological dimension and there are several alternative definitions for that purpose. However, the bulk of them depends on some sort of measurement. Amongst all these fractal dimension, the most prominent ones are the Hausdorff dimension and the Minkowski-Bouligand dimension (also known as box dimension).

These dimensions can be defined in any metric space, and in the specific group theoretical context, the study of the Hausdorff dimension in the setting of profinite groups has attracted considerable attention. Actually, if G is a countably based profinite infinite group, there exists a *filtration series* of G, that is, a family $\{G_n\}_{n\in\mathbb{N}}$ of descending open subgroups which is a neighbourhood system of the identity, i.e. $\bigcap_{n\in\mathbb{N}} G_n = \{1\}$. Such filtration defines a metric on G by letting

$$d(x, y) = \inf \left\{ |G : G_n|^{-1} \mid xy^{-1} \in G_n \right\}.$$

This notion of distance makes G into a metric space. Thus, one can define the Hausdorff and the Minkowski-Bouligand dimensions of a subset $X \subseteq G$ (see Section 3.1 for the precise definitions), which will be denoted, respectively as $\mathrm{hdim}(X)$ and $\mathrm{lbdim}(X)$. There is a unique measure on a profinite group, namely the Haar measure, whereas there might be several non-equivalent metrics; and usually fractal dimensions depend on the metric –or equivalently on the filtration series employed to define it–. Furthermore, for a fixed filtration series $\{G_n\}_{n\in\mathbb{N}}$ we can consider the collection of values $\mathrm{hdim}_{\{G_n\}}(H)$ where H ranges over the closed subgroups of G, that is

$$\mathrm{hspec}_{\{G_n\}}(G) := \left\{\mathrm{hdim}_{\{G_n\}}(H) \mid H \leq_c G\right\},$$

which is called the *Hausdorff spectrum* of G with respect to the filtration series $\{G_n\}_{n\in\mathbb{N}}$. Although we could define the Minkowski-Bouligand spectrum similarly, for natural filtration series we have that $\mathrm{hdim}(H) = \mathrm{lbdim}(H)$ for every closed subgroup $H \leq_c G$ (see upcoming Theorem 3.7). Consequently, in keeping with classical terminology, we will merely use the name "Hausdorff".

It turns out that these spectra might have little or no resemblance as one changes the filtration. For instance, consider the additive pro-p group $\mathbb{Z}_p \oplus \mathbb{Z}_p$. For finitely generated pro-p groups of this kind, there exists a natural filtration series, namely the *p-power filtration series*, given by $G_n = G^{p^n}$. With respect to this series $\mathrm{hspec}_{\{G_n\}}(G) = \{0, 1/2, 1\}$, so it is finite; whilst, by [48, Theorem 1.3] there exists a filtration series $\{H_n\}_{n\in\mathbb{N}}$ such that $\mathrm{hspec}_{\{H_n\}}(G)$ contains the real interval $\left[\frac{1}{p+1}, \frac{p-1}{p+1}\right]$, so, whenever $p > 2$, it is uncountable.

The article [6] of Barnea and Shalev is one of the earliest works concerning Hausdorff dimension in profinite groups, and amid other results, there is shown that $\mathrm{hspec}_{\left\{G^{p^n}\right\}}(G)$ is finite for any p-adic analytic pro-p group G. Nonetheless the converse remains open:

Question 3.1 (cf. [6, Problem 1])**.** Let G be a finitely generated pro-p group such that $\mathrm{hspec}_{\{G^{p^n}\}}(G)$ is finite. Is G p-adic analytic?

It is worthwhile mentioning that the question has positive answer when G is besides a soluble group (see [48, Theorem 1.7]).

However, the p-power filtration series typically can not be used in the setting of profinite R-analytic groups, as G^{p^n} is not normally an open subgroup of a compact R-analytic group G. Nevertheless, those groups possess a canonical filtration series, which depends on the group's analytic structure. By a way of example, in Section 3.2 we study the connection between the analytic dimension of a closed submanifold M and the aforesaid fractal dimensions computed with respect to this natural filtration series, and we obtain the following identity:

$$\mathrm{hdim}(M) = \mathrm{lbdim}(M) = \frac{\max\{\dim_x M \mid x \in M\}}{\dim H}. \tag{3.1}$$

The Hausdorff dimension relative to this filtration series, which is introduced insightfully in Subsection 3.1.3, is called *R-standard Hausdorff dimension*. The corresponding Hausdorff spectrum, the *R-standard Hausdorff spectrum*, is denoted as $\mathrm{hspec}_{\mathrm{st}}$. For p-adic analytic pro-p groups, the finiteness of the Hausdorff spectrum with respect to the p-power filtration can be stated differently:

Theorem 3.2 (cf. [6, Corollary 1.2] and [27, Corollary 3.4]). *Let G be a compact p-adic analytic group. Then $\mathrm{hspec}_{\mathrm{st}}(G)$ is finite and rational.*

Nevertheless, the situation is radically dissimilar when the base ring is distinct from a finitely generated extension of $\mathbb{Z}_p$ (recall that an R-analytic group is p-adic analytic if and only if R is a finitely generated ring extension of $\mathbb{Z}_p$). In this chapter, we shall mostly restrict to the case $R = \mathbb{F}_p[[t]]$, and the main findings of our investigation can be summarised as follows:

Theorem 3.3. *Let G be a compact $\mathbb{F}_p[[t]]$-analytic group.*

(i) *The standard Hausdorff spectrum of G contains the real interval* $[0, {}^{1}/_{\dim G}]$.

(ii) *If G is soluble, then* $\mathrm{hspec}_{\mathrm{st}}(G) = [0, 1]$.

These suggest that the standard spectrum of an R-analytic group might be sufficient to isolate p-adic analytic groups, as the prior results are consonant with the next conjecture:

Conjecture 3.4. Let G be a compact R-analytic group such that $\mathrm{hspec}_{\mathrm{st}}(G)$ is finite. Then G is p-adic analytic.

3.1 Hausdorff and box dimension

In this section, we briefly describe the above-mentioned fractal dimensions. Furthermore, we collect their basic properties and some preliminary results, focusing chiefly on the setting of profinite groups.

3.1.1 Basic definitions and properties

Let us shortly present the fractal dimensions alluded to throughout the previous introductory section. Let (M, d) be a metric space, let $X \subseteq M$ and let δ and z be positive numbers. We define

$$\mathcal{H}^z_\delta(X) := \inf \sum_{n=1}^{\infty} \mathrm{diam}(U_n)^z,$$

where $\{U_n\}_{n\in\mathbb{N}}$ is a δ-*covering*, namely a covering of X consisting of sets of diameter at most δ, and the infimum is taken over all those coverings. Observe that the limit

$$\mathcal{H}^z(X) := \lim_{\delta\to 0} \mathcal{H}^z_\delta(X)$$

exists, since $\mathcal{H}^z_\delta(X)$ is non-decreasing as δ tends to zero. Besides, $\mathcal{H}^z$ is an outer measure (see [28, Proposition 11.17]), called the *z-Hausdorff measure* in M. The following result holds:

Lemma 3.5 (cf. [26, Section 3.2]). *Suppose that $\mathcal{H}^s(X) < \infty$ and $t \geq s$. Then $\mathcal{H}^t(X) = 0$.*

Accordingly, we can define the *Hausdorff dimension* of X with respect to the metric d as

$$\mathrm{hdim}_d(X) := \inf\{s \mid \mathcal{H}^s(X) = 0\} = \sup\{s \mid \mathcal{H}^s(X) = \infty\}$$

–over profinite groups, the metric depends on a filtration series $\{G_n\}_{n\in\mathbb{N}}$, and consequently, we will use the notation $\mathrm{hdim}_{\{G_n\}}$–.

It is straightforward to verify that the Hausdorff dimension is

- *monotone*, that is, $\mathrm{hdim}_d(X) \leq \mathrm{hdim}_d(Y)$ whenever $X \subseteq Y$ and
- *countably stable*, that is, $\mathrm{hdim}_d(\cup_{n\in\mathbb{N}} X_n) = \sup_{n\in\mathbb{N}} \mathrm{hdim}_d(X_n)$

(compare with [26, pp. 48-49]). Furthermore, we highlight the following property:

Proposition 3.6 (cf. [26, Proposition 3.3]). *Let $f\colon (M_1, d_1) \to (M_2, d_2)$ be a* bi-Lipschitz *map between metric spaces, i.e. there exist two positive constants $C, c \in \mathbb{R}_{\geq 0}$ such that*

$$c \cdot d_1(x, y) \leq d_2(f(x), f(y)) \leq C \cdot d_1(x, y) \ \forall x, y \in M_1.$$

Then $\mathrm{hdim}_{d_2}(f(X)) = \mathrm{hdim}_{d_1}(X)$ for all $X \subseteq M_1$. In particular, isometries preserve Hausdorff dimension.

The other fractal dimension we will treat with is the Minkowski-Bouligand dimension or the (lower) box dimension. Let $N_\delta(X)$ be the minimal number of sets of diameter at most δ that are required to cover X, and define respectively the *lower box dimension* and the *upper box dimension* (albeit we will mainly focus on the former) as follows:

$$\mathrm{lbdim}_d(X) := \liminf_{\delta\to 0^+} \frac{\log N_\delta(X)}{-\log \delta} \text{ and } \mathrm{ubdim}_d(X) := \limsup_{\delta\to 0^+} \frac{\log N_\delta(X)}{-\log \delta}.$$

Observe that these expressions are independent of the base to which we take the logarithm. When these two values coincide, and therefore the underlying sequence of real numbers is convergent, that common value is referred to as $\mathrm{bdim}_d(X)$, the *box dimension* of X.

In the context of countably based profinite groups, we can obtain a purely group theoretical formula for the above expressions. Indeed, in a profinite group G, the sole values that take the metric defined by using the filtration series $\{G_n\}_{n\in\mathbb{N}}$ are $|G : G_n|^{-1}$, and the ball of center x and radius $\delta = |G : G_n|^{-1}$ is simply the coclass xG_n. Thus, for every $X \subseteq G$ we have $N_\delta(X) = |XG_n : G_n|$ (this expression stands for the number of cosets of the form xG_n for some $x \in X$), and consequently the preceding definitions can be rewritten as

$$\mathrm{lbdim}_{\{G_n\}}(X) = \liminf_{n\to\infty} \frac{\log |XG_n : G_n|}{\log |G : G_n|}$$

and

$$\mathrm{ubdim}_{\{G_n\}}(X) = \limsup_{n\to\infty} \frac{\log |XG_n : G_n|}{\log |G : G_n|}$$

–we will substitute the subscript with $\{G_n\}$, as the metric is completely determined by the filtration series–. Moreover, it easily follows from these definitions that both box dimensions are monotone and bi-Lipschitz invariant (see [26, Proposition 2.5]). In addition, the upper box dimension is *finitely stable*, that is, $\mathrm{ubdim}(X \cup Y) = \max\{\mathrm{ubdim}(X), \mathrm{ubdim}(Y)\}$. However, the lower box dimension might not have this property.

In his pioneering work [1], Abercrombrie proved that for closed subgroups –and some filtration series– the Hausdorff and the lower box dimension coincide. In other words,

Theorem 3.7 (cf. [6, Theorem 2.4])**.** *Let G be a countably based profinite group with normal filtration series $\{G_n\}_{n\in\mathbb{N}}$, that is, $G_n \trianglelefteq G$ for all $n \in \mathbb{N}$. For every closed subgroup $H \leq_c G$ we have*

$$\mathrm{hdim}_{\{G_n\}}(H) = \mathrm{lbdim}_{\{G_n\}}(H) = \liminf_{n\to\infty} \frac{\log |HG_n : G_n|}{\log |G : G_n|}. \tag{3.2}$$

In fact, in the referred work the author proved that $\mathrm{lbdim}(H) \leq \mathrm{hdim}(H)$, as the other inequality is true in any metric space. In accordance with mathematical literature, we will use the name Hausdorff dimension inasmuch as we will mainly be concerned about the dimension of closed subgroups.

3.1.2 Formulae: subgroups and quotients

Throughout this chapter, relating the Hausdorff dimension of a countably based profinite group to that of its subgroups and quotients will be of vital importance. Therefore, it is sometimes convenient to use the notation $\operatorname{hdim}^G_{\{G_n\}}$ to emphasize that the dimension, with respect to the filtration series $\{G_n\}_{n\in\mathbb{N}}$, is calculated within the group G.

Lemma 3.8 (cf. [48, Lemma 5.3]). *Let G be a countably based profinite group, $\{G_n\}_{n\in\mathbb{N}}$ a normal filtration series of G and $H \le_c G$ a closed subgroup whose Hausdorff dimension is given by a proper limit. Then*

$$\operatorname{hdim}^G_{\{G_n\}}(K) = \operatorname{hdim}^G_{\{G_n\}}(H)\operatorname{hdim}^H_{\{H\cap G_n\}}(K)$$

for all $K \le_c H$.

Remark 3.9. The Hausdorff dimension of H above being a proper limit means that

$$\operatorname{hdim}_{\{G_n\}}(H) = \lim_{n\to\infty} \frac{\log|HG_n : G_n|}{\log|G : G_n|}.$$

Proof. A simple computation shows that

$$\begin{aligned}\operatorname{hdim}^G_{\{G_n\}}(K) &= \liminf_{n\to\infty} \frac{\log|K : K\cap G_n|}{\log|G : G_n|}\\ &= \lim_{n\to\infty}\frac{\log|H : H\cap G_n|}{\log|G : G_n|}\liminf_{n\to\infty}\frac{\log|K : K\cap H\cap G_n|}{\log|H : H\cap G_n|}\\ &= \operatorname{hdim}^G_{\{G_n\}}(H)\operatorname{hdim}^H_{\{H\cap G_n\}}(K).\end{aligned}$$

□

Moreover, for quotients of countably based profinite groups we have the following result:

Lemma 3.10 (cf. [47, Lemma 2.2]). *Let G be a countably based profinite group, $\{G_n\}_{n\in\mathbb{N}}$ a normal filtration series of G and $N \trianglelefteq_c G$ a closed normal subgroup. Assume that the Hausdorff dimension of N is given by a proper limit. Then for every subgroup $H \le_c G$ containing N one has*

$$\operatorname{hdim}^G_{\{G_n\}}(H) = \big(1-\operatorname{hdim}^G_{\{G_n\}}(N)\big)\operatorname{hdim}^{G/N}_{\{G_nN/N\}}(H/N) + \operatorname{hdim}^G_{\{G_n\}}(N).$$

Proof. We observe that

$$\begin{aligned}\frac{\log|HG_n : NG_n|}{\log|G : G_n|} &= \frac{\log|G : NG_n|}{\log|G : G_n|} \cdot \frac{\log|HG_n : NG_n|}{\log|G : NG_n|} \\ &= \frac{\log|G : G_n| - \log|NG_n : G_n|}{\log|G : G_n|} \cdot \frac{\log|HG_n : NG_n|}{\log|G : NG_n|} \\ &= \left(1 - \frac{\log|NG_n : G_n|}{\log|G : G_n|}\right) \frac{\log|HG_n : NG_n|}{\log|G : NG_n|}.\end{aligned}$$

Therefore, since $\mathrm{hdim}^G_{\{G_n\}}(N) = \eta$ is given by a proper limit

$$\begin{aligned}\mathrm{hdim}^G_{\{G_n\}}(H) &= \liminf_{n\to\infty} \frac{\log|HG_n : G_n|}{\log|G : G_n|} \\ &= \liminf_{n\to\infty} \left(\frac{\log|HG_n : NG_n|}{\log|G : G_n|} + \frac{\log|NG_n : G_n|}{\log|G : G_n|}\right) \\ &= \liminf_{n\to\infty} \left(\left(1 - \frac{\log|NG_n : G_n|}{\log|G : G_n|}\right) \frac{\log|HG_n : NG_n|}{\log|G : NG_n|}\right) + \eta \\ &= (1-\eta) \liminf_{n\to\infty} \left(\frac{\log|HG_n/N : NG_n/N|}{\log|G/N : NG_n/N|}\right) + \eta \\ &= (1-\eta)\, \mathrm{hdim}^{G/N}_{\{NG_n/N\}}(H/N) + \eta,\end{aligned}$$

as required. □

Corollary 3.11. *Let G be a countably based profinite group with normal filtration series $\{G_n\}_{n\in\mathbb{N}}$ and let $N \trianglelefteq G$ be a finite normal subgroup. Then*

$$\mathrm{hspec}_{\{G_n\}}(G) = \mathrm{hspec}_{\{G_nN/N\}}(G/N).$$

Proof. Since $\mathrm{hdim}^G_{\{G_n\}}(N) = 0$ is given by a proper limit, the inclusion

$$\mathrm{hspec}_{\{G_nN/N\}}(G/N) \subseteq \mathrm{hspec}_{\{G_n\}}(G)$$

is a direct consequence of the Correspondence Theorem and Lemma 3.10.

For the converse, consider $\eta \in \mathrm{hspec}_{\{G_n\}}(G)$; then there exists $H \leq_c G$ such that $\mathrm{hdim}^G_{\{G_n\}}(H) = \eta$. Since N is finite and the right multiplication is an isometry by Lemma 3.10 one has

$$\begin{aligned}\mathrm{hdim}^G_{\{G_n\}}(H) &= \mathrm{hdim}^G_{\{G_n\}}\left(\bigcup_{n\in N} Hn\right) \\ &= \mathrm{hdim}^G_{\{G_n\}}(HN) = \mathrm{hdim}^G_{\{G_nN/N\}}(HN/N),\end{aligned}$$

as required. □

Finally, the combination of the above results yields the following corollary.

Corollary 3.12. *Let G be a countably based profinite group, $\{G_n\}_{n\in\mathbb{N}}$ a normal filtration series and let $N \trianglelefteq K \leq G$ be closed subgroups such that* $\operatorname{hdim}^G_{\{G_n\}}(N) = \eta$ *and* $\operatorname{hdim}^G_{\{G_n\}}(K) = \kappa$ *are given by proper limits. If* $\operatorname{hspec}_{\left\{\frac{(K\cap G_n)N}{N}\right\}}(K/N) = [0,1]$ *then* $[\eta,\kappa] \subseteq \operatorname{hspec}_{\{G_n\}}(G)$.

Proof. Firstly, by Lemma 3.8 it follows that $\operatorname{hdim}^K_{\{K\cap G_n\}}(N) = \eta/\kappa$, and using the Correspondence Theorem and Lemma 3.10 we obtain

$$[\eta/\kappa, 1] = \left\{(1-\eta/\kappa)\alpha + \eta/\kappa \,\middle|\, \alpha \in \operatorname{hspec}_{\left\{\frac{(K\cap G_n)N}{N}\right\}}(K/N)\right\} \subseteq \operatorname{hspec}_{\{K\cap G_n\}}(K).$$

By another application of Lemma 3.8, one concludes $[\eta,\kappa] \subseteq \operatorname{hspec}_{\{G_n\}}(G)$. □

We conclude this subsection by stating the following result, due to Klopsch, Thillaisundaram and Zugadi-Reizabal in [48], that will be of utility to find profinite groups with full Hausdorff spectrum:

Theorem 3.13 (cf. [48, Theorem 5.4])**.** *Let G be a countably based pro-p group and let $\{G_n\}_{n\in\mathbb{N}}$ be a normal filtration series. Suppose that every finitely generated closed subgroup $H \leq_c G$ satisfies* $\operatorname{hdim}_{\{G_n\}}(H) = 0$. *Then* $\operatorname{hspec}_{\{G_n\}}(G) = [0,1]$.

3.1.3 R-standard Hausdorff dimension

In the context of compact R-analytic groups a natural filtration is available. Indeed, let G be a compact R-analytic group of dimension d and let (S,ϕ) be an open R-standard subgroup of level N. As already presented in (1.6), the *R-standard filtration series* induced by S is the filtration series $\{S_n\}_{n\in\mathbb{N}}$ defined as

$$S_n := \phi^{-1}\left(\left(\mathfrak{m}^{N+n}\right)^{(d)}\right), \ \forall n \in \mathbb{N}_0.$$

These are obviously open subgroups and, by virtue of the Krull Intersection Theorem (loc. cit.) an R-standard filtration series is indeed a filtration series. Furthermore, from (1.5) one has that $S_n \trianglelefteq S$ for every $n \in \mathbb{N}$, and thus formula (3.2) holds for R-standard groups with the above filtration.

Because of the dependence of hdim on the chosen filtration we should not assume *a priori* that the Hausdorff dimension (resp. lower box dimension) of a subgroup of a compact R-analytic group is the same when computed with respect to two different R-standard filtrations. To prove this actual independence, we start by recalling this consequence of (1.7):

Remark 3.14. The *Hilbert function* of $(R, \mathfrak{m})$ is defined as $H\colon \mathbb{N}_0 \to \mathbb{N}$, $n \mapsto \dim_{R/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1})$, and for large enough n it coincides with a polynomial $p(n)$ of degree $\dim_{\mathrm{Krull}}(R) - 1$, called the *Hilbert polynomial* of R (cf. [25, Chapter 6, Theorem C]). Hence, according to the Euler-Maclaurin formula the sum $\sum_{i=1}^{n-1} p(i)$ is asymptotically equivalent to a polynomial $f(n)$ of degree $\dim_{\mathrm{Krull}}(R)$, i.e. their ratio tends to 1 as n tends to infinity.

Let q be the size of the residue field $R/\mathfrak{m}$ and let (S, ϕ) be an R-standard group of dimension d and level N. In view of (1.7),

$$\log_q |S : S_n| = d \sum_{i=N}^{N+n-1} H(i)$$

is asymptotically equivalent to $df(n)$.

The following result shows that the lower box dimension is independent of the standard filtration chosen.

Lemma 3.15 (cf. [27, Theorem 3.1])**.** *Let G be a compact R-analytic group and let (S, ϕ) and (T, ψ) be two open R-standard subgroups of G. For every $X \subseteq G$ we have that*

$$\mathrm{lbdim}_{\{S_n\}}(X) = \mathrm{lbdim}_{\{T_n\}}(X).$$

Proof. Let us denote by $N(S)$ and $N(T)$ respectively the levels of S and T. Firstly, we shall prove the existence of two integers $a, b \in \mathbb{N}$ such that for every integer n satisfying $n - b \in \mathbb{N}$

$$S_{n+a} \leq T_n \leq S_{n-b}. \tag{3.3}$$

Indeed, since the R-analytic map $\psi \circ \phi^{-1}$ is convergent in $\phi(S \cap T) \subseteq \left(\mathfrak{m}^{N(S)}\right)^{(d)}$ and since $\psi \circ \phi^{-1}(\mathbf{0}) = \mathbf{0}$, according to (1.1) there exists $L \geq N(S)$ such that

$$\psi \circ \phi^{-1}\left(\left(\mathfrak{m}^{L+n}\right)^{(d)}\right) \subseteq \left(\mathfrak{m}^{n}\right)^{(d)}$$

for any $n \in \mathbb{N}$. Hence, by setting $a = L - N(S) + N(T)$, we obtain that $S_{n+a} \leq T_n$. Arguing similarly with $\phi \circ \psi^{-1}$ we obtain (3.3). Consequently,

$$\begin{aligned}
\mathrm{lbdim}_{\{T_n\}}(X) &= \liminf_{n\to\infty} \frac{\log |XT_n : T_n|}{\log |G : T_n|} \\
&\leq \liminf_{n\to\infty} \frac{\log |XS_{n+a} : S_{n+a}|}{\log |G : S_{n+a}|} \cdot \frac{\log |G : S_{n+a}|}{\log |G : T_n|} \\
&= \liminf_{n\to\infty} \frac{\log |XS_{n+a} : S_{n+a}|}{\log |G : S_{n+a}|} \cdot \frac{\log |G : S_{n+a}|}{\log |G : S_{n+a}| - \log |T_n : S_{n+a}|} \\
&= \mathrm{lbdim}_{\{S_n\}}(X),
\end{aligned}$$

using in the ultimate equality that

$$\lim_{n\to\infty} \frac{\log|G : S_{n+a}|}{\log|G : S_{n+a}| - \log|T_n : S_{n+a}|} = 1.$$

Indeed, according to the previous remark

$$\log|T_n : S_{n+a}| \leq \log|S_{n-b} : S_{n+a}| = \sum_{i=N+n-b}^{N+n+a-1} H(i).$$

Hence, for large enough n the right hand side term is the sum of $a+b$ polynomials of degree $\dim_{\mathrm{Krull}}(R) - 1$, while $\log|G : S_{n+a}| = \log|G : S| + \log|S : S_{n+a}|$ is asymptotically equivalent to a polynomial of degree $\dim_{\mathrm{Krull}}(R)$. We finish the proof by swapping S and T. □

This allows us to define the *standard* or *R-standard lower box dimension*, denoted $\mathrm{lbdim}_{\mathrm{st}}$, and the *R-standard upper box dimension*, denoted $\mathrm{ubdim}_{\mathrm{st}}$, disregarding the chosen standard filtration. Besides, it is worth noting as an aside that when R is a PID the analogue of Lemma 3.15 can be proved for the Hausdorff dimension.

Lemma 3.16. *Let R be pro-p domain that is a PID, let G be a compact R-analytic group and let (S,ϕ) and (T,ψ) be two open R-standard subgroups of G. For every $X \subseteq G$, we have*

$$\mathrm{hdim}_{\{S_n\}}(X) = \mathrm{hdim}_{\{T_n\}}(X).$$

Proof. Let q be the size of the residue field $R/\mathfrak{m}$ and $d = \dim G$. Since R is a PID, $|S_n : S_{n+1}| = |T_n : T_{n+1}| = q^d$ for every $n \in \mathbb{N}$. From (3.3) there exist two integers $a, b \in \mathbb{N}$ such that $S_{n+a} \leq T_n \leq S_{n-b}$ for every integer n such that $n - b \in \mathbb{N}$. Thus,

$$|G : T_n|^{-1} \geq |G : S_{n+a}|^{-1} = q^{-d(a+b)}|G : S_{n-b}|^{-1}.$$

Hence, if we denote by δ_S and δ_T the distances in G induced respectively by the filtration series $\{S_n\}_{n\in\mathbb{N}}$ and $\{T_n\}_{n\in\mathbb{N}}$, then

$$\delta_T(x,y) \geq q^{-d(a+b)}\delta_S(x,y).$$

By swapping S and T we obtain the existence of a constant $C > 0$ such that $C \cdot \delta_S(x,y) \geq \delta_T(x,y)$. Hence, the identity map between the metric spaces (G,δ_S) and (G,δ_T) is bi-Lipschitz, and the result follows by Proposition 3.6. □

An R-standard filtration $\{S_n\}_{n\in\mathbb{N}}$ defines a Hausdorff dimension in both G and the open R-standard subgroup S. In keeping with previous notation, these dimensions are denoted respectively as $\mathrm{hdim}^G_{\{S_n\}}$ and $\mathrm{hdim}^S_{\{S_n\}}$.

Lemma 3.17. *Let G be a compact R-analytic group with open R-standard subgroup (S, ϕ).*

(i) *Let $H \leq_c G$ and $U \subseteq_o H$, then* $\mathrm{hdim}^G_{\{S_n\}}(H) = \mathrm{hdim}^G_{\{S_n\}}(U)$.

(ii) *Let $X \subseteq S$, then* $\mathrm{hdim}^S_{\{S_n\}}(X) = \mathrm{hdim}^G_{\{S_n\}}(X)$.

In particular, $\mathrm{hdim}^G_{\{S_n\}}(H) = \mathrm{hdim}^S_{\{S_n\}}(H \cap S)$ *for every closed subgroup $H \leq_c G$.*

Remark 3.18. The preceding result is also true for the *upper* box dimension, with exactly the same proof. For the *lower* box dimension we only have (ii).

Proof. (i) Since H is a compact subgroup, being a closed subset of the compact group G, there exist finitely many $g_i \in G$ such that $H = \cup_{i=1}^r g_i U$. Thus, by the finite stability of the Hausdorff dimension and since left multiplication maps are an isometries, using Proposition 3.6 we have

$$\mathrm{hdim}^G_{\{S_n\}}(H) = \mathrm{hdim}^G_{\{S_n\}}\left(\cup_{i=1}^r g_i U\right) = \max_{i=1,\dots,r} \mathrm{hdim}^G_{\{S_n\}}(g_i U) = \mathrm{hdim}^G_{\{S_n\}}(U).$$

(ii) Let δ_G and δ_S be the metrics induced by the filtration $\{S_n\}_{n\in\mathbb{N}}$ in G and in S respectively. Then $\delta_G(x,y) = |G : S|^{-1}\delta_S(x,y)$ and so the inclusion map from (S, δ_S) to (G, δ_G) is bi-Lipschitz, and the identity follows by Proposition 3.6.

For the final conclusion, since $H \cap S$ is an open subgroup of H, applying (i) and (ii), we deduce that

$$\mathrm{hdim}^G_{\{S_n\}}(H) = \mathrm{hdim}^G_{\{S_n\}}(H \cap S) = \mathrm{hdim}^S_{\{S_n\}}(H \cap S),$$

as required. □

Corollary 3.19. *Let G be a compact R-analytic group and let (S, ϕ) and (T, ψ) be two open R-standard subgroups of G. For every $H \leq_c G$ we have*

$$\mathrm{hdim}_{\{S_n\}}(H) = \mathrm{hdim}_{\{T_n\}}(H).$$

Proof. In view of Theorem 3.7, Lemma 3.15 and the fact that Lemma 3.17(ii) holds for the lower box dimension, we have:

$$\begin{aligned}\mathrm{hdim}^{G}_{\{S_n\}}(H) &= \mathrm{hdim}^{S}_{\{S_n\}}(H\cap S\cap T) = \mathrm{lbdim}^{S}_{\{S_n\}}(H\cap S\cap T)\\ &= \mathrm{lbdim}^{G}_{\{S_n\}}(H\cap S\cap T) = \mathrm{lbdim}^{G}_{\{T_n\}}(H\cap S\cap T)\\ &= \mathrm{lbdim}^{T}_{\{T_n\}}(H\cap S\cap T) = \mathrm{hdim}^{T}_{\{T_n\}}(H\cap S\cap T)\\ &= \mathrm{hdim}^{G}_{\{T_n\}}(H).\end{aligned}$$

□

Therefore, we define *standard* or *R-standard Hausdorff dimension* of a closed subgroup $H \leq_c G$, which will be denoted $\mathrm{hdim}_{\mathrm{st}}(H)$, as $\mathrm{hdim}_{\{S_n\}}(H)$ where $\{S_n\}_{n\in\mathbb{N}}$ is *any* R-standard filtration. Accordingly, the *standard* or *R-standard Hausdorff spectrum* of G is

$$\mathrm{hspec}_{\mathrm{st}}(G) := \{\mathrm{hdim}_{\mathrm{st}}(H) \mid H \leq_c G\}.$$

Moreover, we have the following immediate consequence of Lemma 3.17:

Corollary 3.20. *Let G be a compact R-analytic group with an open R-standard subgroup (S,ϕ). Then* $\mathrm{hspec}_{\mathrm{st}}(G) = \mathrm{hspec}_{\mathrm{st}}(S)$.

Thus, in order to study the standard Hausdorff spectrum of a compact R-analytic group we can assume that the original group G is itself an R-standard group.

Finally, we shall study the standard Hausdorff dimension of analytic subgroups and quotients. The following lemma relates $\mathrm{hdim}^{H}_{\mathrm{st}}$ with the Hausdorff dimension on H induced in the natural way by an R-standard filtration $\{S_n\}_{n\in\mathbb{N}}$ of G, namely $\mathrm{hdim}^{H}_{\{H\cap S_n\}}$.

Lemma 3.21. *Let G be a compact R-analytic group and let H be an R-analytic subgroup of G. Then $\mathrm{hdim}^{H}_{\{H\cap S_n\}}(K) = \mathrm{hdim}^{H}_{\mathrm{st}}(K)$ for all $K \leq_c H$ and any R-standard filtration $\{S_n\}_{n\in\mathbb{N}}$ of G.*

Proof. Firstly, let $\{S_n\}_{n\in\mathbb{N}}$ and $\{T_n\}_{n\in\mathbb{N}}$ be two R-standard filtrations of G. By Lemma 3.8 and Corollary 3.19, it is straightforward that

$$\mathrm{hdim}^{H}_{\{H\cap S_n\}}(K) = \mathrm{hdim}^{H}_{\{H\cap T_n\}}(K),\ \forall K \leq_c H. \tag{3.4}$$

Secondly, we shall show that there exists an open R-standard subgroup S of G such that $\{H\cap S_n\}_{n\in\mathbb{N}}$ is an R-standard filtration for H. Then, by (3.4), for any other R-standard filtration $\{T_n\}_{n\in\mathbb{N}}$ of G, we would have that

$$\mathrm{hdim}^{H}_{\{H\cap T_n\}}(K) = \mathrm{hdim}^{H}_{\{H\cap S_n\}}(K) = \mathrm{hdim}^{H}_{\mathrm{st}}(K)$$

for all $K \leq_c H$, as desired.

Let $d = \dim G$ and $k = \dim H$, since H is an R-analytic subgroup, by Definition 1.48 there exists an R-chart (U, ϕ) of 1 in G such that

$$\begin{aligned}\phi(H \cap U) &= \left\{(x_1, \ldots, x_d) \in \left(\mathfrak{m}^N\right)^{(d)} \middle|\ x_{k+1} = \cdots = x_d = 0\right\} \\ &= \left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)},\end{aligned}$$

for some $N \geq 1$, and $\phi(1) = \mathbf{0}$. Furthermore, by Lemma 1.21 there exists an open R-standard subgroup S of G, of level $L \geq N$, contained in U and whose corresponding homeomorphism is $\phi|_S$. Then

$$\begin{aligned}\phi\left(H \cap S\right) &= \phi\left(S\right) \cap \phi(H \cap U) \\ &= \left(\mathfrak{m}^L\right)^{(d)} \cap \left(\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)}\right) = \left(\mathfrak{m}^L\right)^{(k)} \times \{0\}^{(d-k)}.\end{aligned}$$

Therefore, if $\mathrm{pr}\colon (\mathfrak{m}^L)^{(k)} \times \{0\}^{(d-k)} \to (\mathfrak{m}^L)^{(k)}$ is the natural projection, then $(H \cap S, \psi)$, where $\psi = \mathrm{pr} \circ \phi|_{H \cap S}$, is an open R-standard subgroup of H. Moreover,

$$\psi\left(H \cap S_n\right) = \mathrm{pr}\left(\phi(H \cap U) \cap \phi\left(S_n\right)\right) = \left(\mathfrak{m}^{L+n}\right)^{(k)},$$

and one concludes that $\{H \cap S_n\}_{n \in \mathbb{N}}$ is an R-standard filtration of H. □

For quotients, if G is a compact R-analytic group and $N \trianglelefteq G$ is a normal R-analytic subgroup, then G/N is likewise a compact R-analytic group, according to Proposition 1.59. Hence, we shall relate the standard spectrum of the group and the spectrum of its analytic quotients.

Lemma 3.22. *Let G be a compact R-analytic group, $\{S_n\}_{n \in \mathbb{N}}$ an R-standard filtration series of G and $N \trianglelefteq G$ a normal R-analytic subgroup of G. Then*

$$\mathrm{hdim}_{\mathrm{st}}(H) = \mathrm{hdim}_{\left\{\frac{S_n N}{N}\right\}}(H),$$

for every $H \leq_c G/N$.

Proof. Let us fix some notation: let $d = \dim G$ and $e = \dim G/N$, let $\pi\colon G \to G/N$ be natural epimorphism and let $\mathrm{pr}\colon \mathfrak{m}^{(d)} \to \mathfrak{m}^{(e)}$ be the projection onto the last e coordinates.

Firstly, if $\{S_n\}_{n\in\mathbb{N}}$ and $\{T_n\}_{n\in\mathbb{N}}$ are two R-standard filtrations of G, exactly as in the proof of Lemma 3.15 and by virtue of Theorem 3.7 it can be seen that for any $H \leq_c G/N$,

$$\operatorname{hdim}_{\left\{\frac{S_nN}{N}\right\}}(H) = \operatorname{lbdim}_{\left\{\frac{S_nN}{N}\right\}}(H) = \operatorname{lbdim}_{\left\{\frac{T_nN}{N}\right\}}(H) = \operatorname{hdim}_{\left\{\frac{T_nN}{N}\right\}}(H)$$

Consequently, it suffices to find an open R-standard subgroup S of G such that $\{S_nN/N\}_{n\in\mathbb{N}}$ is an R-standard filtration of G/N. According to Lemma 1.58, there exists an R-chart (U,ϕ) of 1 in G adapted to N, that is, $\operatorname{pr}\circ\phi(x) = \operatorname{pr}\circ\phi(y)$ if and only if $xy^{-1} \in N$. Suppose further that $\phi(1) = \mathbf{0}$. Moreover, from Lemma 1.21 there exists an open R-standard subgroup S, of level L, contained in U and with homeomorphism $\phi|_S$. Let $\sigma\colon \pi(S) \to S$ be a continuous section such that $\sigma(1N) = 1$, which exists by [64, Proposition 2.2.2]. Then $\pi(S)$ is an R-standard subgroup of G/N, with level L, dimension e and homeomorphism $\psi = \operatorname{pr}\circ\phi\circ\sigma$. Note that since (U,ϕ) is an adapted R-chart, the definition of ψ is independent of the selected section and $\psi(S_nN/N) = \operatorname{pr}\circ\phi(S_n) = (\mathfrak{m}^{L+n})^{(e)}$, so $\{S_nN/N\}_{n\in\mathbb{N}}$ is an R-standard filtration of G/N. □

3.2 Hausdorff dimension of submanifolds

In [27], the authors study the relationship between the analytic dimension and the Hausdorff dimension of an R-analytic subgroup. This investigation relies mainly upon the following particular case.

Lemma 3.23. *Let $(R,\mathfrak{m})$ be pro-p domain. The box dimension of $\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)}$ in the additive R-analytic group $\mathfrak{m}^{(d)}$ with respect to the filtration series $\left\{(\mathfrak{m}^n)^{(d)}\right\}_{n\in\mathbb{N}}$ is a proper limit of value k/d.*

Remark. The Hausdorff dimension of $\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)}$ is also k/d, considering that it is a closed subgroup (see Theorem 3.7).

Proof. Let q be the size of the residue field $R/\mathfrak{m}$, and recall, from Remark 3.14, that

$$\left|\mathfrak{m}^{(d)} : (\mathfrak{m}^n)^{(d)}\right| = q^{d\sum_{i=1}^{n-1} H(i)},$$

where H stands for the Hilbert function of R, so whenever $n > N$,

$$\left|\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)} : \left(\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)}\right) \cap (\mathfrak{m}^n)^{(d)}\right|$$
$$= \left|\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)} : (\mathfrak{m}^n)^{(k)} \times \{0\}^{(d-k)}\right| = q^{k\sum_{i=N}^{n-1} H(i)}.$$

Therefore,

$$\begin{aligned}
&\operatorname{lbdim}_{\mathrm{st}}\left(\left(\mathfrak{m}^N\right)^{(k)}\times\{0\}^{(d-k)}\right)\\
&\qquad=\liminf_{n\to\infty}\frac{\log_q\left|\left(\mathfrak{m}^N\right)^{(k)}\times\{0\}^{(d-k)}:\left(\left(\mathfrak{m}^N\right)^{(k)}\times\{0\}^{(d-k)}\right)\cap\left(\mathfrak{m}^n\right)^{(d)}\right|}{\log_q\left|\mathfrak{m}^{(d)}:\left(\mathfrak{m}^n\right)^{(d)}\right|}\\
&\qquad=\liminf_{n\to\infty}\frac{k\sum_{i=N}^{n-1}H(i)}{d\sum_{i=1}^{n-1}H(i)}=\frac{k}{d}.
\end{aligned}$$

Using in the last equality that

$$\lim_{n\to\infty}\frac{H(1)+\cdots+H(n-1)}{H(N)+\cdots+H(n-1)}=1+\lim_{n\to\infty}\frac{H(1)+\cdots+H(N-1)}{H(N)+\cdots+H(n-1)}=1,$$

as in the central term the denominator is asymptotically equivalent to a non-constant polynomial and the numerator is a constant value. As a consequence, this dimension is given by a proper limit. □

In the next two corollaries we prove (3.1).

Corollary 3.24. *Let G be a compact R-analytic group and let M be a closed submanifold. Then*

$$\operatorname{ubdim}_{\mathrm{st}} M=\frac{\max\{\dim_x M\mid x\in M\}}{\dim G},$$

and this dimension is a proper limit.

Remark 3.25. In particular, we can consider $\operatorname{bdim}_{\mathrm{st}} M$, the *standard box dimension* of a closed submanifold M.

Proof. Let d be the analytic dimension of G. According to Definition 1.48, for each $x\in M$ there exist $U_x\subseteq_o M$ containing x and a regular R-chart (V_x,ψ) such that $U_x\subseteq V_x$ and

$$\psi(U_x)=\{(x_1,\dots,x_d)\in\psi(V_x)\mid x_{k+1}=\cdots=x_d=0\},$$

where $k=\dim_x M$. Moreover, passing to an open subset, we can assume without lose generality, that $V_x=xS$ for some open R-standard subgroup S with homeomorphism $\phi\colon S\to\left(\mathfrak{m}^N\right)^{(d)}$, $y\mapsto\psi(xy)$. Consequently,

$$\phi\left(x^{-1}U_x\right)=\psi(U_x)=\left(\mathfrak{m}^N\right)^{(k)}\times\{0\}^{(d-k)}. \tag{3.5}$$

Besides, Lemma 1.20 can be rewritten as ϕ being an isometry between the group S with the metric induced by the R-standard filtration $\{S_n\}_{n\in\mathbb{N}}$ and the additive group $\left(\mathfrak{m}^N\right)^{(d)}$ with the metric induced by the filtration $\left\{\left(\mathfrak{m}^{N+n}\right)^{(d)}\right\}_{n\in\mathbb{N}}$. Therefore, since left multiplication is an isometry and according to Remark 3.18, we have

$$\mathrm{ubdim}^G_{\{S_n\}}(U_x) = \mathrm{ubdim}^G_{\{S_n\}}(x^{-1}U_x) = \mathrm{ubdim}^S_{\{S_n\}}(x^{-1}U_x).$$

Further, using successively Proposition 3.6 and Lemma 3.23, and in view of (3.5) we obtain

$$\mathrm{ubdim}^G_{\{S_n\}}(U_x) = \mathrm{ubdim}^S_{\{S_n\}}(x^{-1}U_x) = \mathrm{ubdim}^{\left(\mathfrak{m}^N\right)^{(d)}}_{\left\{\left(\mathfrak{m}^{N+n}\right)^{(d)}\right\}}(\phi(x^{-1}U_x)) = \frac{\dim_x M}{\dim G}, \tag{3.6}$$

and this dimension is a proper limit. On the one hand, by the monotonicity,

$$\frac{\dim_x M}{\dim G} = \mathrm{ubdim}_{\mathrm{st}}(U_x) \leq \mathrm{ubdim}_{\mathrm{st}}(M) \tag{3.7}$$

for all $x \in M$. On the other hand, since M is compact, being a closed subset of the compact group G, there exist finitely many $x_i \in M$ such that $M = \cup_{i\in I} U_{x_i}$. Hence, according to (3.6) and using the finite stability,

$$\mathrm{ubdim}_{\mathrm{st}}(M) = \mathrm{ubdim}_{\mathrm{st}}\left(\cup_{i\in I} U_{x_i}\right) = \max_{i\in I} \mathrm{ubdim}_{\mathrm{st}}(U_{x_i}) \leq \frac{\max\{\dim_x M \mid x \in M\}}{\dim G}, \tag{3.8}$$

which, together with (3.7), yields the result for the upper box dimension. □

The analogue result for the Hausdorff dimension is a bit more involved.

Corollary 3.26. *Let G be a compact R-analytic group, let $\{S_n\}_{n\in\mathbb{N}}$ be an R-standard filtration and let M be a closed submanifold. Then*

$$\mathrm{hdim}_{\{S_n\}}(M) = \frac{\max\{\dim_x M \mid x \in M\}}{\dim G}.$$

Proof. This proof follows the same line as the previous proof, and we keep the notation therein. Particularly, for each $x \in M$ there exist $U_x \subseteq_o M$ and an open R-standard subgroup (S, ϕ) such that $x^{-1}U_x \subseteq S$ and

$$\phi(x^{-1}U_x) = \left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)},$$

where $k = \dim_x M$. Reproducing the arguments of (3.6) verbatim, if δ_S is the distance in G induced by S, we obtain that

$$\operatorname{hdim}_{\delta_S}(U_x) = \frac{\dim_x M}{\dim G}.$$

Let (T, ψ) be another open R-standard subgroup of G and let δ_T be the distance defined in G by using T. On the one hand, from [26, Proposition 3.4] and (3.6),

$$\operatorname{hdim}_{\delta_T}(U_x) \leq \operatorname{ubdim}_{\mathrm{st}}(U_x) = \frac{\dim_x M}{\dim G}.$$

Let us define $\tilde{U}_x = x^{-1}U_x \cap T$, so in particular $\psi(\tilde{U}_x) \subseteq \psi(S \cap T)$. Then

$$\phi(\tilde{U}_x) = \phi(x^{-1}U_x \cap T) = \left(\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)}\right) \cap \phi(S \cap T),$$

and since $\phi(S \cap T)$ is open in $\mathfrak{m}^{(d)}$, there exists $K \in \mathbb{N}$ such that

$$\left(\mathfrak{m}^K\right)^{(k)} \times \{0\}^{(d-k)} \subseteq \phi(\tilde{U}_x).$$

Let us denote by d the distance defined on $\mathfrak{m}^{(d)}$ using the standard filtration series $\left\{(\mathfrak{m}^n)^{(d)}\right\}_{n\in\mathbb{N}}$. Since ϕ and ψ are isometries, then $\phi \circ \psi^{-1}$ is also an isometry from $(\psi(S \cap T), d)$ to $(\phi(S \cap T), d)$. Hence, in view of Lemma 3.17, Proposition 3.6 and the monotonicity,

$$\begin{aligned}\operatorname{hdim}^G_{\delta_T}(U_x) &\geq \operatorname{hdim}^T_{\delta_T}\left(\tilde{U}_x\right) = \operatorname{hdim}^{\mathfrak{m}^{(d)}}_{\delta}\left(\psi(\tilde{U}_x)\right) = \operatorname{hdim}^{\mathfrak{m}^{(d)}}_{\delta}\left(\phi \circ \psi^{-1} \circ \psi(\tilde{U}_x)\right) \\ &= \operatorname{hdim}^{\mathfrak{m}^{(d)}}_{\delta}(\phi(\tilde{U}_x)) \geq \operatorname{hdim}^{\mathfrak{m}^{(d)}}_{\{(\mathfrak{m}^n)^{(d)}\}}\left(\left(\mathfrak{m}^K\right)^{(k)} \times \{0\}^{(d-k)}\right) = k/d,\end{aligned}$$

using Lemma 3.23 in the last equality. Therefore, $\operatorname{hdim}(U_x) = \dim_x M/\dim G$, unregarding the standard filtration chosen. We finish as in (3.7) and (3.8), but replacing the box dimension with the Hausdorff dimension. □

Considering that R-analytic subgroups are closed pure R-analytic submanifolds, we recover the principal result in [27]:

Corollary 3.27 (cf. [27, Main Theorem])**.** *Let G be a compact R-analytic group and let H be an R-analytic subgroup. Then*

$$\operatorname{bdim}_{\mathrm{st}}(H) = \operatorname{hdim}_{\mathrm{st}}(H) = \frac{\dim H}{\dim G}.$$

In particular, both dimensions are a proper limit.

When $R = \mathbb{Z}_p$ every closed subgroup is p-adic analytic (see [24, Theorem 9.6]), so we obtain an alternative proof of Theorem 3.2, and in passing we get a clear-cut expression for standard spectra in this setting. In fact, if G is a d-dimensional compact p-adic analytic group, then

$$\mathrm{hspec}_{\mathrm{st}}(G) \subseteq \left\{0, \frac{1}{d}, \dots, \frac{d-1}{d}, 1\right\}.$$

3.3 Abelian compact R-analytic groups

Henceforth, we will carry on describing the standard spectra of profinite R-analytic groups that are not p-adic analytic, i.e. R is not a finite extension of $\mathbb{Z}_p$. In the first place, we will deal with the abelian case.

Remark. From now on we will work with closed subgroups, and Theorem 3.7 will be implicitly used.

Proposition 3.28. *Let R be a pro-p domain of characteristic p or Krull dimension at least 2, and let (S, ϕ) be an abelian R-standard group. Then* $\mathrm{hspec}_{\mathrm{st}}(S) = [0, 1]$.

Proof. By Theorem 3.13 it suffices to prove that every finitely generated closed subgroup $H \leq_c S$ satisfies $\mathrm{hdim}_{\mathrm{st}}(H) = 0$. Let d be the dimension of S and let $H \leq S$ be a topologically r-generated closed subgroup.

If R has characteristic p, since the group operation in S is given by a formal group law, by (1.3) whenever $x \in S_n$ we have

$$\phi(x^p) \equiv p\phi(x) = \mathbf{0} \mod \left(\mathfrak{m}^{2n}\right)^{(d)},$$

and thus $x^p \equiv 1 \pmod{S_{2n}}$. Therefore S_n/S_{2n} is an elementary abelian p-group.

Since S is abelian, $H/(H \cap S_n)$ is an abelian p-group of exponent p^e where $e \leq \lceil \log_2(n) \rceil$. Moreover, H is topologically r-generated, so $H/(H \cap S_n)$ is r-generated, and thus

$$|H : H \cap S_n| \leq p^{er} \leq p^{\lceil \log_2(n) \rceil r}.$$

According to Remark 3.14, if $|R/\mathfrak{m}| = q = p^c$, then $|S : S_n|$ is asymptotically equivalent to $q^{df(n)}$ where $f(n)$ is a polynomial of degree $\dim_{\mathrm{Krull}}(R)$. Consequently,

$$\mathrm{hdim}_{\mathrm{st}}(H) = \liminf_{n \to \infty} \frac{\log_p |H : H \cap S_n|}{\log_p |S : S_n|} \leq \liminf_{n \to \infty} \frac{r \lceil \log_2(n) \rceil}{cdf(n)} = 0,$$

as desired.

Similarly, if R has Krull dimension of at least 2, by (1.3), whenever $x \in S_n$ we have

$$\phi(x^p) \equiv p\phi(x) \equiv \mathbf{0} \mod \left(\mathfrak{m}^{n+1}\right)^{(d)},$$

so S_n/S_{n+1} is an elementary abelian p-group. Consequently, $H/(H \cap S_n)$ is an r-generated abelian group of exponent p^e, where $e \leq n-1$. Therefore

$$|H : H \cap S_n| \leq p^{re} \leq p^{r(n-1)},$$

so, according to Remark 3.14,

$$\mathrm{hdim}_{\mathrm{st}}(H) = \liminf_{n\to\infty} \frac{\log_p |H : H \cap S_n|}{\log_p |S : S_n|} \leq \liminf_{n\to\infty} \frac{r(n-1)}{cdf(n)} = 0,$$

as $f(n)$ is a polynomial of degree $\dim_{\mathrm{Krull}}(R) \geq 2$. □

Clearly, in view of of Corollary 3.20, this result can be generalised to abelian compact R-analytic groups.

Corollary 3.29. *Let R be a pro-p domain of characteristic p or Krull dimension at least 2. If G is an abelian compact R-analytic group, then* $\mathrm{hspec}_{\mathrm{st}}(G) = [0,1]$.

Furthermore, it is known that any R-standard group of dimension one is abelian (see [35, Theorem 1.6.7]), and we thus have the following:

Corollary 3.30. *Let R be a pro-p domain of characteristic p or Krull dimension at least 2 and let G be a compact R-analytic group of dimension one. Then* $\mathrm{hspec}_{\mathrm{st}}(G) = [0,1]$.

3.4 Compact $\mathbb{F}_p[[t]]$-analytic groups

Section 3.3 invites to surmise that when G is a soluble compact R-analytic group that is not p-adic analytic, its R-standard spectrum is the whole real interval $[0,1]$. The main strategy to prove that would lie in adding successive intervals to the spectrum, using the consecutive abelian quotients of a subnormal series. In fact, we have the following result:

Lemma 3.31. *Let G be a compact R-analytic group and let $N \trianglelefteq K \leq G$ be R-analytic subgroups such that* $\mathrm{hspec}_{\mathrm{st}}(K/N) = [0,1]$. *Then*

$$\left[\frac{\dim N}{\dim G}, \frac{\dim K}{\dim G}\right] = [\mathrm{hdim}_{\mathrm{st}}(N), \mathrm{hdim}_{\mathrm{st}}(K)] \subseteq \mathrm{hspec}_{\mathrm{st}}(G).$$

Proof. The equality is a direct consequence of Corollary 3.27, namely $\operatorname{hdim}_{\mathrm{st}}(H) = {}^{\dim H}/_{\dim G}$ for every analytic subgroup $H \leq G$ and such dimension is given by a proper limit, and the inclusion is straightforward from Corollary 3.12, Lemma 3.21 and Lemma 3.22. □

Thus, we shall establish a useful criterion for finding R-analytic subgroups of a compact R-analytic group. The main obstacle compared with classical Lie theory arises here: it is well known that any closed subgroup of a real (p-adic) Lie group is a real (p-adic) Lie subgroup; nevertheless for R-analytic groups, closeness is a necessary condition (see Lemma 1.57), but not sufficient. For example, the additive group $\mathbb{F}_p[[t]]$ is an $\mathbb{F}_p[[t]]$-analytic group and $\mathbb{F}_p[\![t^2]\!]$ is a closed subgroup with its own $\mathbb{F}_p[[t]]$-analytic group structure. However, those manifold structures are not compatible, so $\mathbb{F}_p[\![t^2]\!]$ is not an $\mathbb{F}_p[[t]]$-analytic subgroup of $\mathbb{F}_p[[t]]$.

Now we will turn to the case when $R = \mathbb{F}_p[[t]]$. The task of finding $\mathbb{F}_p[[t]]$-analytic subgroups can be carried out by using Proposition 1.53, which shows that analytic subsets have a manifold structure over $\mathbb{F}_p[[t]]$. According to the definition therein, a set $X \subseteq M$ is an analytic subset if for each $x \in X$ there exist an open neighbourhood U of x and some $\mathbb{F}_p[[t]]$-analytic functions $f_1, \ldots, f_r$ defined on U (for some $r = r_x$) such that

$$X \cap U = \{y \in U \mid f_i(y) = 0 \ \forall i = 1, \ldots, r\}.$$

We then have:

Theorem 3.32 (cf. [45, Corollary 4.2]). *Let G be an $\mathbb{F}_p[[t]]$-analytic group and let H be both a subgroup of G and an analytic subset of G. Then H is an $\mathbb{F}_p[[t]]$-analytic subgroup of G.*

Let us see some examples of applications of the preceding theorem:

Corollary 3.33. *Let S be an $\mathbb{F}_p[[t]]$-standard group and a in S. Then $Z(S)$ and $C_S(a)$ are $\mathbb{F}_p[[t]]$-analytic subgroups.*

Proof. By the previous theorem it is enough to show that $Z(S)$ and $C_S(a)$ are analytic subsets. The former is proved in [45, Corollary 4.3], while the latter follows the same spirit. Indeed, since S is $\mathbb{F}_p[[t]]$-standard of level say N and dimension say d, then it can be identified with $\left(t^N\right)^{(d)}$. Since the group operation is given by a formal group law, by (1.5) there exist some $g_{i,\alpha} \in \mathbb{F}_p[[t]][[X_1, \ldots, X_d]]$ such that

$$\pi_i\left(y^{-1}ay\right) = a_i + \sum_{|\alpha|\geq 1} g_{i,\alpha}(a) y_1^{\alpha_1} \ldots y_d^{\alpha_d} = a_i + h_i(y)$$

for all y in S, where the map $\pi_i\colon \left(t^N\right)^{(d)} \to \left(t^N\right)$ is the projection to the ith coordinate. Moreover, the maps $h_i(y) = \sum_{|\alpha|\geq 1} g_{i,\alpha}(a) y_1^{\alpha_1} \dots y_d^{\alpha_d}$ are clearly $\mathbb{F}_p[[t]]$-analytic. Therefore

$$\begin{aligned} C_S(a) &= \left\{ y \in S \mid \pi_i\left(y^{-1}ay\right) = a_i \ \forall i = 1, \dots, d \right\} \\ &= \{ y \in S \mid h_i(y) = 0 \ \forall i = 1, \dots, d \}, \end{aligned}$$

and $C_S(a)$ is an analytic subset. □

The second application involves the general linear group $\mathrm{GL}_n(R)$. In addition to the usual topology in $\mathrm{GL}_n(R)$, namely the one induced by the ring topology of R, we also have the so-called Zariski topology, in which the closed subsets are the *affine sets*, i.e. subsets of the form

$$\{ A \in \mathrm{GL}_n(R) \mid f(A) = 0 \ \forall f \in \mathcal{F} \},$$

where $\mathcal{F} \subseteq R[\mathbf{X}]$ is a subset of polynomials in n^2 variables. Note as well that any subgroup $H \leq \mathrm{GL}_n(R)$ can be likewise endowed with both the usual subspace topology or the weaker (polynomial maps are continuous with respect to the $\mathfrak{m}$-adic topology) Zariski topology.

Let us present some general facts concerning the Zariski topology:

Proposition 3.34 (cf. [73, Lemma 5.9 and Theorem 5.11])**.** *Let $H \leq \mathrm{GL}_n(R)$ and let $\mathcal{H}$ be its Zariski closure.*

(i) *Then $\mathcal{H} \leq \mathrm{GL}_n(R)$.*

(ii) *If H is normal in $\mathrm{GL}_n(R)$, so is $\mathcal{H}$.*

(iii) *Suppose that H is nilpotent of class c. Then, $\mathcal{H}$ has a central series of length c consisting of Zariski closed subgroups. In particular, $\mathcal{H}$ is nilpotent of class c.*

(iv *Suppose that H is soluble of length c. Then $\mathcal{H}$ has a subnormal series of length c consisting of Zariski closed subgroups whose quotient groups are abelian. In particular, $\mathcal{H}$ is soluble of length c.*

(iv) *Let K be a subgroup of H such that $|H : K|$ is finite and let $\mathcal{K}$ be the Zariski closure of K in $\mathrm{GL}_n(R)$. Then $\mathcal{K}$ has finite index in $\mathcal{H}$.*

Corollary 3.35. *Let $G \subseteq \mathrm{GL}_n(\mathbb{F}_p[[t]])$ be a linear $\mathbb{F}_p[[t]]$-analytic group and let $\mathcal{H}$ be a Zariski closed subgroup of $\mathrm{GL}_n(\mathbb{F}_p[[t]])$. Then $\mathcal{H} \cap G$ is an $\mathbb{F}_p[[t]]$-analytic subgroup of G.*

Proof. Since $\mathcal{H}$ is closed in the Zariski topology, it is an affine set, that is, there exists a subset $\mathcal{F}$ of $\mathbb{F}_p[[t]][\mathbf{X}]$, where $\mathbf{X}$ is a tuple of n^2 variables, such that

$$\mathcal{H} = \{A \in \mathrm{GL}_n(\mathbb{F}_p[[t]]) \mid f(A) = 0 \ \forall f \in \mathcal{F}\}.$$

But since $\mathbb{F}_p[[t]][\mathbf{X}]$ is Noetherian we can assume $\mathcal{F}$ to be finite, and thus

$$\mathcal{H} \cap G = \{A \in G \mid f(A) = 0 \ \forall f \in \mathcal{F}\}$$

is an analytic subset, so it is an $\mathbb{F}_p[[t]]$-analytic subgroup by Theorem 3.32. □

We are now in a position to prove part of Theorem 3.3 by using the previous results:

Theorem 3.36. *Let G be a soluble compact $\mathbb{F}_p[[t]]$-analytic group. Then,*

$$\mathrm{hspec}_{\mathrm{st}}(G) = [0, 1].$$

Proof. By Corollary 3.20, we can assume without loss of generality that G is $\mathbb{F}_p[[t]]$-standard. We first prove the theorem for the case when G is linear over $\mathbb{F}_p[[t]]$, that is, $G \subseteq \mathrm{GL}_n(\mathbb{F}_p[[t]])$. Let $\mathcal{G}$ be the Zariski closure of G in $\mathrm{GL}_n(\mathbb{F}_p[[t]])$. According to Proposition 3.34, $\mathcal{G}$ is a soluble group, and there exists a subnormal series

$$\mathcal{G} = \mathcal{H}_1 \trianglerighteq \mathcal{H}_2 \trianglerighteq \cdots \trianglerighteq \mathcal{H}_{\ell-1} \trianglerighteq \mathcal{H}_\ell = \{1\}$$

consisting of Zariski closed subgroups whose quotient groups are all abelian. Then

$$G = \mathcal{H}_1 \cap G \trianglerighteq \mathcal{H}_2 \cap G \trianglerighteq \cdots \trianglerighteq \mathcal{H}_{\ell-1} \cap G \trianglerighteq \mathcal{H}_\ell \cap G = \{1\}$$

is a soluble series of G given by $\mathbb{F}_p[[t]]$-analytic subgroups by Corollary 3.35.

Denote $H_i = \mathcal{H}_i \cap G$. Since each H_i is an $\mathbb{F}_p[[t]]$-analytic subgroup of G then H_{i-1}/H_i is a compact abelian $\mathbb{F}_p[[t]]$-analytic group for all $i \in \{2, \ldots, \ell\}$, so by Corollary 3.29 it follows that $\mathrm{hspec}_{\mathrm{st}}(H_i/H_{i-1}) = [0, 1]$. Hence by Lemma 3.31 one has that $[\mathrm{hdim}_{\mathrm{st}}(H_i), \mathrm{hdim}_{\mathrm{st}}(H_{i-1})] \subseteq \mathrm{hspec}_{\mathrm{st}}(G)$ for all $i \in \{2, \ldots, \ell\}$, and thus $\mathrm{hspec}_{\mathrm{st}}(G) = [0, 1]$.

Let us finally turn to the general case. By Corollary 3.33, $Z(G)$ is an abelian $\mathbb{F}_p[[t]]$-analytic subgroup of G and thus by Corollary 3.29 and Lemma 3.31

$$[0, \mathrm{hdim}_{\mathrm{st}} Z(G)] \subseteq \mathrm{hspec}_{\mathrm{st}}(G).$$

Moreover, by Propositions 1.59 and 2.3 one has that $G/Z(G)$ is a compact soluble $\mathbb{F}_p[[t]]$-analytic group that is linear over $\mathbb{F}_p[[t]]$. Hence, according to Lemmata 3.21 and 3.22,

$$\mathrm{hspec}_{\{S_n Z(G)/Z(G)\}}(G/Z(G)) = \mathrm{hspec}_{\mathrm{st}}\,(G/Z(G)) = [0,1],$$

and so by Corollary 3.12

$$[\mathrm{hdim}_{\mathrm{st}}\, Z(G), 1] \subseteq \mathrm{hspec}_{\mathrm{st}}(G),$$

thus obtaining the whole interval in the spectrum. □

More generally, a suitable way to find an interval in the $\mathbb{F}_p[[t]]$-standard Hausdorff spectrum of a compact $\mathbb{F}_p[[t]]$-analytic group G is looking for a soluble $\mathbb{F}_p[[t]]$-analytic subgroup. This search will rely heavily on the topological analogue of the Tits alternative. But we first observe the following:

Lemma 3.37. *Let G be an $\mathbb{F}_p[[t]]$-standard group. Suppose that either*

(i) *$Z(G)$ is infinite or*

(ii) *G contains an element x of infinite order.*

Then $[0, {}^{1}/_{\dim G}] \subseteq \mathrm{hspec}(G)$.

Proof. Under the first hypothesis, by Corollary 3.33, $Z(G)$ is an abelian infinite $\mathbb{F}_p[[t]]$-analytic subgroup. Similarly, under the second hypothesis $Z(C_G(x))$ is an abelian $\mathbb{F}_p[[t]]$-analytic subgroup which is infinite, because $\langle x\rangle \leq Z(C_G(x))$. In both cases, there exists an infinite abelian $\mathbb{F}_p[[t]]$-analytic subgroup $H \leq G$. Since G is compact, H has strictly positive analytic dimension, and according to Corollary 3.29 the $\mathbb{F}_p[[t]]$-standard spectrum of H is the whole interval $[0,1]$. Finally, by Lemma 3.31, $[0, {}^{\dim H}/_{\dim G}]$ is contained in the $\mathbb{F}_p[[t]]$-standard spectrum. □

Theorem 3.38. *Let G be a compact $\mathbb{F}_p[[t]]$-analytic group. Then,*

$$[0, {}^{1}/_{\dim G}] \subseteq \mathrm{hspec}_{\mathrm{st}}(G).$$

Proof. We can assume, in view of Corollary 3.20, that G is R-standard. Firstly, observe that when $Z(G)$ is infinite the result follows by Lemma 3.37(i), so we shall deal with the case when $Z(G)$ is finite. But then $G/Z(G)$ is an $\mathbb{F}_p[[t]]$-analytic group of dimension $\dim G$ and according to Corollary 3.11 it follows that

$$\mathrm{hspec}_{\mathrm{st}}(G) = \mathrm{hspec}_{\mathrm{st}}(G/Z(G)).$$

Furthermore, by Proposition 2.3, $G/Z(G)$ is an $\mathbb{F}_p[[t]]$-analytic group that is linear over $\mathbb{F}_p[[t]]$. Hence by the topological Tits alternative (cf. [12, Theorem 1.3]) it follows that $G/Z(G)$ contains either an open soluble subgroup, say H, or contains a dense free subgroup. In the former case, H is a soluble $\mathbb{F}_p[[t]]$-analytic group of dimension $\dim G/Z(G) = \dim G$ and thus

$$\mathrm{hspec}_{\mathrm{st}}(G/Z(G)) = [0,1]\,.$$

In the latter case $G/Z(G)$ contains an element of infinite order and the statement follows by Lemma 3.37(ii). □

3.5 Classical Chevalley groups

The spectrum of a compact $\mathbb{F}_p[[t]]$-analytic group need not be the whole interval $[0,1]$. For instance, consider the *special linear group* $\mathrm{SL}_n(\mathbb{F}_p[[t]])$. It is well-known that $\mathrm{SL}_n(\mathbb{F}_p[[t]])$ is a compact $\mathbb{F}_p[[t]]$-analytic group of dimension n^2-1, containing as an open subgroup the $\mathbb{F}_p[[t]]$-standard group

$$\mathrm{SL}_n^1\left(\mathbb{F}_p[[t]]\right) := \ker\left\{\mathrm{SL}_n(\mathbb{F}_p[[t]] \to \mathrm{SL}_n\left(\mathbb{F}_p[[t]]/t\mathbb{F}_p[[t]]\right)\right\}.$$

In [6, Corollary 1.5], the standard spectrum of $\mathrm{SL}_2(\mathbb{F}_p[[t]])$ is completely established when $p > 2$, to wit

$$\mathrm{hspec}_{\mathrm{st}}\left(\mathrm{SL}_2(\mathbb{F}_p[[t]])\right) = [0, 2/3] \cup \{1\}.$$

Moreover, in [6, Theorem 1.4] it is proved that when $p > 2$,

$$\mathrm{hspec}_{\mathrm{st}}\left(\mathrm{SL}_n(\mathbb{F}_p[[t]])\right) \cap \left(1 - \frac{1}{n+1}, 1\right) = \varnothing,$$

and 1 is an isolated point of the spectrum thereof. We will provide further examples of compact $\mathbb{F}_p[[t]]$-analytic groups whose spectrum is not the whole interval, by proving an analogous result for the other classical Chevalley groups. For that purpose, we will follow the same techniques already used in [6] and work in the corresponding graded Lie algebra. We start with a brief summary of those matrix groups. For basic definitions regarding root systems and a comprehensive analysis on the topic we refer to [15]. Let R be a general pro-p domain.

- The Chevalley group over R associated to a root system of type A_n ($n \geq 1$) is $\mathrm{SL}_{n+1}(R)$.

- A root system of type B_n ($n \geq 2$) defines the odd *special orthogonal group*

$$\mathrm{SO}_{2n+1}(R) := \left\{ A \in \mathrm{M}_{2n+1}(R) \;\middle|\; A^t K_{2n+1} A = K_{2n+1} \right\},$$

where $K_n = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix} \in \mathrm{M}_n(R)$, which is an R-analytic group of dimension $n(2n+1)$.

- A root system of type C_n ($n \geq 3$) defines the *symplectic group*

$$\mathrm{Sp}_{2n}(R) := \left\{ A \in \mathrm{M}_{2n}(R) \mid A^t J_{2n} A = J_{2n} \right\},$$

where $J_{2n} = \begin{pmatrix} 0 & K_n \\ -K_n & 0 \end{pmatrix}$, which is an R-analytic group of dimension $n(2n+1)$.

- A root system of type D_n ($n \geq 4$) defines the even *special orthogonal group*

$$\mathrm{SO}_{2n}(R) := \left\{ A \in \mathrm{M}_{2n}(R) \mid A^t K_{2n} A = K_{2n} \right\},$$

which is an R-analytic group of dimension $n(2n-1)$.

All those groups are compact, being closed subsets of the compact space $\mathrm{M}_n(R) \cong R^{(n^2)}$. That is, classical Chevalley groups over R are actually compact R-analytic groups. Furthermore, the following result describes their associated Lie algebras.

Theorem 3.39 (cf. [24, Exercise 13.11(iii)])**.** *Let X_n be a root system of type A_n ($n \geq 1$), B_n ($n \geq 2$), C_n ($n \geq 3$) or D_n ($n \geq 4$). Let $G(R)$ be the Chevalley group associated to X_n over a pro-p domain R.*

(i) *If $X_n = A_n$, there exists an open R-standard group S such that*

$$\mathcal{L}(S) \cong \mathfrak{sl}_{n+1}(R) = \{ A \in \mathrm{M}_{n+1}(R) \mid \mathrm{tr}(A) = 0 \}.$$

(ii) *If $X_n = B_n$, there exists an open R-standard subgroup S such that*

$$\mathcal{L}(S) \cong \mathfrak{so}_{2n+1}(R) = \left\{ A \in \mathrm{M}_{2n+1}(R) \mid A^t = -A \right\}.$$

(iii) *If* $X_n = C_n$*, there exists an open* R*-standard subgroup* S *such that*

$$\mathcal{L}(S) \cong \mathfrak{sp}_{2n}(R) = \left\{ A \in \mathrm{M}_{2n}(R) \mid J_{2n}A + A^t J_{2n} = 0 \right\},$$

where J_{2n} *is defined as before.*

(iv) *If* $X_n = D_n$*, there exists an open* R*-standard subgroup* S *such that*

$$\mathcal{L}(S) \cong \mathfrak{so}_{2n}(R) = \left\{ A \in \mathrm{M}_{2n}(R) \mid A^t = -A \right\}.$$

Here $\mathcal{L}(S)$ *is an abbreviation for the Lie algebra associated to* S *(compare with Section 1.4).*

We start by presenting the construction in [54, Definition 2.9]. Given an R-standard group (S, ϕ) and the corresponding R-standard filtration $\{S_n\}_{n\in\mathbb{N}}$ we define the *graded Lie* $R/\mathfrak{m}$*-algebra* $\mathrm{gr}\mathcal{L}(S) = \bigoplus_{n\geq 0} S_n/S_{n+1}$, which is so with the Lie bracket obtained extending by bilinearity the rule

$$[xS_{n+1}, yS_{m+1}]_{\mathrm{gr}\mathcal{L}(S)} := [x, y]S_{n+m+1}$$

(the right-hand side brackets stand for the group commutator in S).

On the one hand, $[,]_{\mathrm{gr}\mathcal{L}(S)}$ is a Lie bracket. Indeed, from (1.5),

$$\phi([x, y]) = \mathbf{B}(\phi(x), \phi(y)) - \mathbf{B}(\phi(y), \phi(x)) \mod \phi(S_{n+m+1}),$$

and therefore $[\cdot, \cdot]_{\mathrm{gr}\mathcal{L}(S)}$ satisfies Jacobi's identity by virtue of Lemma 1.25. On the other hand, whenever $x \in S_n$ and $y \in S_m$, then $[x, y] \in S_{n+m}$, so the above-defined algebra is graded over the natural numbers. Finally, let q be the cardinality of $R/\mathfrak{m}$. Since each S_n/S_{n+1} is an $R/\mathfrak{m}$-vector space, $\mathrm{gr}\mathcal{L}(S)$ is a graded $\mathbb{F}_q$-Lie algebra.

Any closed subgroup $H \leq_c S$ defines a graded subalgebra of $\mathrm{gr}\mathcal{L}(S)$, which by abuse of notation we will denote by $\mathrm{gr}\mathcal{L}(H)$ and is given by

$$\mathrm{gr}\mathcal{L}(H) := \bigoplus_{n\geq 0} \frac{(H \cap S_n)S_{n+1}}{S_{n+1}}.$$

Although every closed subgroup defines a graded subalgebra, there might be graded subalgebras that do not arise in this way.

For a general graded $\mathbb{F}_q$-algebra $L = \oplus_{n\geq 0} L_n$ and a graded $\mathbb{F}_q$-subalgebra $K = \bigoplus_{n\geq 0} K_n$, we can define the *Hausdorff density* of K as

$$\mathrm{hD}(K) := \liminf_{n\to\infty} \frac{\sum_{0\leq m\leq n} \dim_{\mathbb{F}_q} K_m}{\sum_{0\leq m\leq n} \dim_{\mathbb{F}_q} L_m}.$$

Clearly, in view of the preceding definitions for any closed subgroup H we have that $\mathrm{hD}(\mathrm{gr}\mathcal{L}(H)) = \mathrm{hdim}_{\{S_n\}}(H)$. Indeed,

$$\begin{aligned}
\mathrm{hD}(\mathrm{gr}\mathcal{L}(H)) &= \liminf_{n\to\infty} \frac{\sum_{m\leq n} \dim_{\mathbb{F}_q} {}^{(H\cap S_m)S_{m+1}}/_{S_{m+1}}}{\sum_{m\leq n} \dim_{\mathbb{F}_q} {}^{S_m}/_{S_{m+1}}} \\
&= \liminf_{n\to\infty} \frac{\sum_{m\leq n} \log_q |H\cap S_m : H\cap S_{m+1}|}{\sum_{m\leq n} \log_q |S_m : S_m + 1|} \\
&= \liminf_{n\to\infty} \frac{\log_q \prod_{m\leq n} |H\cap S_m : H\cap S_{m+1}|}{\log_q \prod_{m\leq n} |S_m : S_{m+1}|} \\
&= \liminf_{n\to\infty} \frac{\log_q |H : H\cap S_{n+1}|}{\log_q |S : S_{n+1}|} = \mathrm{hdim}_{\{S_n\}}(H). \qquad (3.9)
\end{aligned}$$

Therefore, for our purpose, proving that 1 is an isolated point in the spectrum, we should look into the maximal graded subalgebras of $\mathrm{gr}\mathcal{L}(S)$. More specifically, we shall prove that whenever their Hausdorff density is not exactly 1, then it is bounded above by $1 - {}^1/_d$. We must concentrate on maximal subalgebras of infinite codimension, as the ones with finite codimension have Hausdorff density equal to 1. For that we have the following structural description of $\mathrm{gr}\mathcal{L}(S)$:

Lemma 3.40 (cf. [24, Proposition 13.27] and [54, Remark 2.10(5)])**.** *As an* $R/\mathfrak{m}$-*Lie algebra* $\mathrm{gr}\mathcal{L}(S)$ *is isomorphic to* $\mathcal{L}_0(S) \otimes_{R/\mathfrak{m}} \mathrm{gr}\,\mathfrak{m}$, *where*

$$\mathcal{L}_0(S) = {}^{\mathcal{L}(S)}/_{\mathfrak{m}\mathcal{L}(S)} \quad \textit{and} \quad \mathrm{gr}\,\mathfrak{m} = \bigoplus_{n\geq 1} \mathfrak{m}^n/\mathfrak{m}^{n+1}.$$

Remark. Observe that $\mathrm{gr}\,\mathfrak{m}$ is the maximal ideal of the graded ring $\mathrm{gr}\,R$.

Proof. Observe that $\mathcal{L}_0(S)$ is nothing but the $R/\mathfrak{m}$-Lie algebra $(R/\mathfrak{m})^{(d)}$ whose Lie bracket is $[\cdot,\cdot]_{\mathcal{L}}$ reduced modulo $\mathfrak{m}$, and so as $R/\mathfrak{m}$-vector spaces $\mathrm{gr}\mathcal{L}(S)$ and $\mathcal{L}_0 \otimes_{R/\mathfrak{m}} \mathrm{gr}\,\mathfrak{m}$ are isomorphic one to another. Finally, since $[\cdot,\cdot]_{\mathcal{L}}$ is bilinear and the tensor product preserves bilinearity, the preceding isomorphism is indeed an $R/\mathfrak{m}$-Lie algebra isomorphism. $\square$

In particular, when $R = \mathbb{F}_p[[t]]$, the resulting graded algebra is $\mathcal{L}_0(S) \otimes_{\mathbb{F}_p} t\mathbb{F}_p[t]$. Let now F be a field and $\mathcal{G}$ a finite dimensional perfect, i.e. $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$, F-Lie algebra. We can consider the infinite dimensional F-Lie algebra $\mathcal{G} \otimes_F tF[t]$ with Lie bracket defined as $[A \otimes t^n, B \otimes t^m] := [A, B]_{\mathcal{G}} \otimes t^{n+m}$ on elementary tensors. We note the following:

Lemma 3.41. *Let $\mathcal{L} = \mathcal{G} \otimes_F tF[t]$ be as above. Then, any graded F-subalgebra of infinite codimension is contained in a graded F-subalgebra of infinite codimension maximal with respect to that property.*

Proof. Firstly, $\mathcal{L}$ is a finitely generated F-Lie algebra. In fact, let $\{x_1, \ldots, x_m\}$ be a generating set of $\mathcal{G}$, then $S = \{x_1, \ldots, x_m, x_1 \otimes t, \ldots, x_m \otimes t\}$ generates $\mathcal{L}$. Indeed, $\langle S \rangle_F$, the Lie algebra generated by S, contains $\mathcal{G}$ and $\mathcal{G} \otimes t$. Hence, if we assume by induction that $\langle S \rangle_F$ contains $\mathcal{G} \otimes t^{n-1}$, then, since $\mathcal{G}$ is perfect

$$\mathcal{G} \otimes t^n = [\mathcal{G}, \mathcal{G}] \otimes t^n = \left[\mathcal{G} \otimes t^{n-1}, \mathcal{G} \otimes t\right] \subseteq \langle S \rangle_F.$$

Finally, the result follows by Zorn's Lemma. We shall show that every chain of graded F-subalgebras of infinite codimension has an upper element. Indeed, let $\{H_i\}_{i \in I}$ be a totally ordered subset of the partially ordered set consisting of graded F-Lie algebras of infinite codimension, and consider $H = \cup_{i \in I} H_i$, which is a graded F-subalgebra of $\mathcal{L}$. Suppose by contradiction that H has finite codimension in $\mathcal{L}$, and so that it is a finitely generated F-algebra. Assume that $H = \langle h_1, \ldots, h_r \rangle_F$, then there exists an $i_0 \in I$ such that $h_k \in H_{i_0}$ for all $k \in \{1, \ldots, r\}$ and so $H = H_{i_0}$ has infinite codimension in $\mathcal{L}$, which is a contradiction. Hence $\{H_i\}_{i \in I}$ has an upper bound with respect to inclusion, which concludes the proof. □

If one requires central simplicity, we have the following result bounding the Hausdorff density of graded subalgebras that are maximal with respect to having infinite codimension.

Theorem 3.42 (cf. [6, Corollary 5.3])**.** *Let $\mathcal{G}$ be a central simple algebra over a field F and let $\mathcal{L} = \mathcal{G} \otimes_F tF[t]$. The density of a graded subalgebra that is maximal with respect to having infinite codimension is either $1/q$, where q is a prime, or $\dim_F \mathcal{H} / \dim_F \mathcal{G}$, where $\mathcal{H}$ is a maximal graded subalgebra of $\mathcal{G}$.*

Remark 3.43. Recall that a finite dimensional algebra $\mathcal{G}$ over a field F is called central simple when it is simple and its *centroid*, i.e.

$$\mathrm{Cent}(\mathcal{G}) = \{f \in \mathrm{End}_F(\mathcal{G}) \mid f([x, y]) = [f(x), y] \ \forall x, y \in \mathcal{G}\},$$

is isomorphic to F. Nevertheless, if F is finite, the assumption of the previous theorem can be weakened to only requiring that $\mathcal{G}$ is simple. Indeed, the previous theorem is proved as [7, Theorem 4.1] and in the remark after [7, Theorem 1.1] it is pointed out that when F is finite, simplicity of $\mathcal{G}$ is enough.

Now, we gather all these results.

Corollary 3.44. *Let X_n be a root system of type A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$ or D_n $(n \geq 4)$, let G be the classical Chevalley group associated to X_n over $\mathbb{F}_p[[t]]$ and $L(Q)$ the Lie algebra associated to that root system over an arbitrary ring Q(see Theorem 3.39). If $L(\mathbb{F}_p)$ is simple, then*

$$\mathrm{hspec}_{\mathrm{st}}(G) \cap \left(1 - \frac{1}{\dim G}, 1\right) = \varnothing.$$

Proof. On the one hand, by Theorem 3.39 it follows that G contains an open $\mathbb{F}_p[[t]]$-standard subgroup, say S, such that the Lie algebra associated to S is $L(\mathbb{F}_p[[t]])$. Furthermore, by Lemma 3.40 there exists an isomorphism $\mathrm{gr}\mathcal{L}(S) \cong \mathcal{L}_0 \otimes_{\mathbb{F}_p} \mathrm{gr}\,\mathfrak{m}$ as $\mathbb{F}_p$-vector spaces where

$$\mathcal{L}_0 = L(\mathbb{F}_p[[t]])/tL(\mathbb{F}_p[[t]]) \cong L(\mathbb{F}_p) \quad \text{ and } \quad \mathrm{gr}\,\mathfrak{m} = \bigoplus_{n\geq 1} (t^n) \,/\, \left(t^{n+1}\right).$$

That is, $\mathrm{gr}\mathcal{L}(S) \cong L(\mathbb{F}_p) \otimes_{\mathbb{F}_p} t\mathbb{F}_p[t]$.

On the other hand, let $H \leq_c S$ be a closed subgroup such that $\mathrm{hdim}_{\mathrm{st}}(H) < 1$. Then $|S : H|$ is infinite and so $\mathrm{gr}\mathcal{L}(H)$ has infinite codimension in $\mathrm{gr}\mathcal{L}(S)$. Since $L(\mathbb{F}_p)$ is simple, according to Lemma 3.41 we know that $\mathrm{gr}\mathcal{L}(H)$ is contained in a graded subalgebra of $\mathrm{gr}\mathcal{L}(S)$, say $\mathcal{M}$, maximal with respect to having infinite codimension. Hence by Theorem 3.42, Remark 3.43 and identity (3.9), we have

$$\begin{aligned} \mathrm{hdim}_{\mathrm{st}}(H) &= \mathrm{hD}(\mathrm{gr}\mathcal{L}(H)) \leq \mathrm{hD}(\mathcal{M}) \\ &\leq \max\left\{\frac{1}{2}, \frac{\dim_{\mathbb{F}_p} \mathcal{H}}{\dim_{\mathbb{F}_p} L(\mathbb{F}_p)} \;\middle|\; \mathcal{H} \text{ maximal subalgebra of } L(\mathbb{F}_p)\right\} \\ &\leq 1 - \frac{1}{\dim_{\mathbb{F}_p} L(\mathbb{F}_p)} = 1 - \frac{1}{\dim S}, \end{aligned}$$

because $\dim S = \dim G = \dim_{\mathbb{F}_p} L(\mathbb{F}_p)$ (compare with Theorem 3.39). Therefore, the result follows since $\mathrm{hspec}_{\mathrm{st}}(G) = \mathrm{hspec}_{\mathrm{st}}(S)$, by virtue of Corollary 3.20. $\square$

Finally, the algebras $\mathfrak{so}_n(F)$ and $\mathfrak{sp}_{2n}(F)$ over a field of positive characteristic p have been thoroughly studied. When $p = 2$ none of them is simple (see [70, Section 4.4]), but when $p \geq 3$ it is well known that $\mathfrak{so}_n(F)$ $(n \geq 5)$ and $\mathfrak{sp}_{2n}(F)$ $(n \geq 2)$ are simple algebras (see [70, pg. 181 and Section 4.4]). Hence we deduce that:

Corollary 3.45. *Let $p \geq 3$ and assume that G is either $\mathrm{SO}_n(\mathbb{F}_p[[t]])$ $(n \geq 5)$ or $\mathrm{Sp}_{2n}(\mathbb{F}_p[[t]])$ $(n \geq 2)$. Then*

$$\mathrm{hspec}_{\mathrm{st}}(G) \cap \left(1 - \frac{1}{\dim G}, 1\right) = \varnothing.$$

3.6 Notes

We refer to Falconer [26] for an account of fractal geometry over Euclidean spaces. Although the results therein are for $\mathbb{R}^{(n)}$, they hold in full generality in any other metric space, and the citations throughout the chapter are valid under this observation.

Section 3.2 follows [27], and the majority of the ideas, bar some technical difficulties derived from the fact that our submanifolds are not typically subgroups, appear in the article above.

The contents of Sections 3.3 to 3.5 constitute chiefly original research material –due in and have been published in the joint article [30] with González-Sánchez. Nonetheless, as mentioned before, the techniques of Section 3.5 reproduce those of Barnea and Shalev [6]. Furthermore, it is noticeable that the principal, albeit not unique, obstruction to generalising Theorem 3.3 to *all* compact R-analytic groups that are not p-adic analytic is the absence of an effective criterion to identify analytic subgroups, i.e. the absence of a result comparable with Proposition 1.53.

Finally, during the PhD the author and de las Heras have published the dissemination article [36], wrote in Basque about fractal dimension.

In the beginning was the Word

John 1:1

4

Words in compact R-analytic groups

A *WORD* in k variables $x_1, \ldots, x_k$ is an element $w(x_1, \ldots, x_k)$ of the free group in k generators $F(x_1, \ldots, x_k)$, that is,

$$x_{i_1}^{\varepsilon_{i_1}} \ldots x_{i_m}^{\varepsilon_{i_m}},$$

where $m \in \mathbb{N}_0,\ \ i_j \in \{1, , \ldots, k\}$ and $\varepsilon_{i_j} = \pm 1$. The special case where $m = 0$ is referred to as the *empty word*.

For a fixed group G, a word $w(x_1, \ldots, x_k)$ can alternatively be viewed as a map $w \colon G^{(k)} \to G$, called *word map*, where $w(g_1, \ldots, g_k)$ is the element of G that we obtain by substituting g_i with x_i for all i. We shall think of words in this second way, i.e. two words are *equivalent* if they define the same word map in every group G. It is straightforward that two words are equivalent if and only if they adopt the same reduced form in the free group.

Definition 4.1. Let G be a group and let $w(x_1, \ldots, x_k)$ be a word. The set of *w-values* in G is

$$w\{G\} := \{w(g_1, \ldots, g_k) \mid g_i \in G\},$$

or, equivalently, $\operatorname{im} w$ if we regard w as a word map in G.

Lemma 4.2. *Let G be an abelian group and let w be a word. Then $w\{G\}$ is a subgroup.*

Proof. Since the group is abelian,

$$w(g_1, \ldots, g_k) \cdot w(h_1, \ldots, h_k) = w(g_1h_1, \ldots, g_kh_k) \in w\{G\},$$

and $w(g_1, \ldots, g_k)^{-1} = w\left(g_1^{-1}, \ldots, g_k^{-1}\right) \in w\{G\}$ for all $g_i, h_i \in G$. □

Nevertheless, typically $w\{G\}$ is not a subgroup of G, and the *verbal subgroup* of w in G is defined as

$$w(G) := \langle w\{G\} \rangle \tag{4.1}$$

–even though throughout this manuscript we will keep the definition (4.1), in the context of topological groups, $w(G)$ is customarily defined as the topological closure of the abstract subgroup generated by the set of w-values–. We will use the following fact without citation:

Lemma 4.3. *Let w be a word and let $\varphi\colon G \to H$ be a group homomorphism. Then $\varphi(w(G)) = w\left(\varphi(G)\right)$. In particular, $w(G) \operatorname{char} G$ and whenever $N \trianglelefteq G$*

$$w\left(G/N\right) = w(G)N/N.$$

Proof. Note that $\varphi\left(w(g_1, \ldots, g_k)\right) = w\left(\varphi(g_1), \ldots, \varphi(g_k)\right)$ for all $g_i \in G$. □

Definition 4.4. The *marginal subgroup* of w in G is

$$w^*(G) := \left\{ g \in G \;\middle|\; \begin{array}{c} w(x_1, \ldots, x_k) = w\left(x_1, \ldots, x_{i-1}, x_ig, x_{i+1}, \ldots, x_k\right) \\ \forall i \in \{1, \ldots, k\}, \forall x_j \in G \end{array} \right\}.$$

Furthermore, $H \leq G$ is *marginal* for w when $H \leq w^*(G)$.

In particular, whenever $H \leq G$ is marginal for w, then $w(H) = \{1\}$. Moreover, if $g \in w^*(G)$, then $w(x_1, \ldots, x_k) = w(x_1, \ldots, gx_i, \ldots, x_k)$ for all $i \in \{1, \ldots, k\}$ and for all $x_j \in G$, that is, in the definition, the right and left positions are interchangeable. Indeed, suppose that $g \in G$ satisfies that $w(x_1, \ldots, x_ig, \ldots, x_k) = w(x_1, \ldots, x_k)$ for all $x_j \in G$, then

$$\begin{aligned} w(x_1, \ldots, gx_i, \ldots, x_k) &= w(x_1, \ldots, x_ig^{x_i}, \ldots, x_k) \\ &= w\left(x_1, \ldots, (x_ig)^{x_i}, \ldots, x_k\right) \\ &= w\left(x_1^{x_i^{-1}}, \ldots, x_ig, \ldots, x_k^{x_i^{-1}}\right)^{x_i} \\ &= w\left(x_1^{x_i^{-1}}, \ldots, x_i, \ldots, x_k^{x_i^{-1}}\right)^{x_i} = w(x_1, \ldots, x_i, \ldots, x_k). \end{aligned}$$

Examples 4.5. Let G be a group.

(i) Both G itself and the trivial subgroup $\{1\}$ are verbal subgroups corresponding, respectively, to the word $w(x) = x$ and the empty word.

(ii) One of the most common words is the *commutator* $\gamma_2(x, y) = [x, y] = x^{-1}y^{-1}xy$. Its verbal subgroup is the *derived subgroup* $\gamma_2(G) = G'$ and its marginal subgroup is the centre $\gamma_2^*(G) = Z(G)$.

(iii) *Lower central words* are defined recursively as

$$\gamma_n(x_1, \ldots, x_n) := [\gamma_{n-1}(x_1, \ldots, x_{n-1}), x_n] \ \forall n \geq 3,$$

and *derived words* are defined recursively as $\delta_1(x_1, x_2) := \gamma(x_1, x_2)$ and

$$\delta_n(x_1, \ldots, x_{2^n}) := [\delta_{n-1}(x_1, \ldots, x_{2^{n-1}}), \delta_{n-1}(x_{2^{n-1}+1}, \ldots, x_{2^n})] \ \forall n \geq 2.$$

(iv) The *Burnside word* $w_m(x) = x^m$ defines the verbal subgroup G^m, the subgroup generated by the mth powers of elements of G, and for instance, the marginal subgroup $w_2^*(G)$ consists on all central elements of order dividing 2, that is, $w_2^*(G) = \{g \in Z(G) \mid g^2 = 1\}$.

P. Hall [33] posed several questions regarding the relation between the set of w-values and the corresponding verbal and marginal subgroups. The following definitions will serve to summarise some of those:

Definition 4.6. Let w be a word and let $\mathcal{C}$ be a class of groups.

(i) A word w is *concise* in $\mathcal{C}$ if for every $G \in \mathcal{C}$ we have that $w(G)$ is finite whenever $w\{G\}$ is finite.

(ii) A word w is *robust* in $\mathcal{C}$ if for every $G \in \mathcal{C}$ we have that $w(G)$ is finite whenever $|G : w^*(G)|$ is finite.

Consonantly, w is *concise* (resp. *robust*) if it is concise (resp. robust) in the class of all groups. Likewise, if every word is concise in a group G, we will say that G is *verbally concise.*

In general, since $|w\{G\}| \leq |G : w^*(G)|^k$ (being k the number of variables of the word w), robustness is stronger than conciseness: if w is concise in $\mathcal{C}$, then w is robust in $\mathcal{C}$. However, over residually finite groups, and all the groups we are concerned about are so, both concepts are equivalent:

Lemma 4.7 (cf. [67, Lemma 1.4.1])**.** *Let G be a group and let w be a word.*

(i) *If $|G : w^*(G)|$ is finite, then $w\{G\}$ is finite.*

(ii) *If G is residually finite and $w\{G\}$ is finite, then $|G : w^*(G)|$ is finite.*

One of the original predictions of P. Hall was that all groups were verbally concise. However, this conjecture was refuted almost three decades later by Ivanov [39], by finding a group G and a word w such that $w\{G\}$ has two elements, but $w(G)$ is infinite cyclic. Nonetheless, this counterexample, or the comparable counterexample constructed by Ol'shanskiĭ (see [62, Theorem 39.7]), is not residually finite. This leads us to the following conjecture, proposed by Jaikin-Zapirain [44] and Segal [67]:

Conjecture 4.8 (Conciseness conjecture for residually finite groups)**.** Every word is concise in the class of residually finite groups.

There are few known examples of classes of verbally concise groups. Apart from the obvious examples of abelian (see Lemma 4.2) and periodic (see upcoming Lemma 4.15) groups; in the decade of 1960s, Merzjalkov [57] and Turner-Smith [72] proved, respectively, that linear groups and groups all of whose quotients are residually finite (e.g. virtually nilpotent groups) are verbally concise.

When the set of w-values is infinite, the comparable notion to that of conciseness is *verbal ellipticity*. In order to define it, we will use the following notation: for a subset $X \subseteq G$, we denote by $X^{*\ell}$ the set of all products of at most ℓ elements of X and their inverses.

Definition 4.9. Let w be a word and let G be a group. We say that w is *elliptic* in G, if there exists $\ell \in \mathbb{N}$ such that $w(G) = w\{G\}^{*\ell}$.

The smallest of such integers ℓ is the *verbal width* of w in G. In consonance, a group G is *verbally elliptic* when every word is elliptic in G. Moreover, ellipticity is stronger than conciseness: whenever w is elliptic in all the groups of a class $\mathcal{C}$, then w is also concise in $\mathcal{C}$.

It follows from Lemma 4.2 that abelian groups are verbally elliptic and in them all words have verbal width equal to 1. Aside from them, it is known that linear algebraic groups* (cf. [56]), finitely generated virtually abelian-by-nilpotent

*by a *linear algebraic group* we mean a Zariski closed subgroup of $\mathrm{GL}_n(K)$ for some $n \in \mathbb{N}$ and an algebraically closed field K.

groups (cf. [29] and [71]) or, directly related to the topic of this thesis, compact p-adic analytic groups (cf. [44]) are verbally elliptic.

Nonetheless, there are natural examples of non-verbally elliptic groups. For instance, Roman'kov [66] presented a finitely generated soluble pro-p group where the second derived word $\delta_2(x_1, \dots, x_4) = [[x_1, x_2], [x_3, x_4]]$ has infinite width.

Regarding profinite groups, verbal width is related to whether the corresponding verbal subgroup is closed.

Proposition 4.10. *Let G be a compact Hausdorff topological group and let w be a word. Then w is elliptic in G if and only if $w(G)$ is closed.*

Proof. For the *only if* implication, note that for every integer n the set $w\{G\}^{*n}$ is closed, as it is the continuous image of a compact set. Thus, if ℓ is the verbal width of w, then $w(G) = w\{G\}^{*\ell}$ is closed.

For the *if*, note that

$$w(G) = \bigcup_{n\in\mathbb{N}} w\{G\}^{*n}$$

where each $w\{G\}^{*n}$ is closed. Therefore, since $w(G)$ is a compact Hausdorff topological space, by the *Baire Category Theorem* (cf. [59, Theorem 48.2]), there exists an integer m such that $w\{G\}^{*m}$ has non-empty interior in $w(G)$, i.e. it contains a non-empty open subset $U \subseteq_o w(G)$. Hence,

$$w(G) = \bigcup_{g\in w(G)} gU,$$

and by the compactness of $w(G)$

$$w(G) = \bigcup_{i=1}^{r} g_i U,$$

for some elements $g_1, \dots, g_r \in w(G)$. Take $k \in \mathbb{N}$ such that $g_i \in w\{G\}^{*k}$ for all $i \in \{1, \dots, r\}$, then

$$w(G) = \bigcup_{i=1}^{r} g_i U \subseteq w\{G\}^{*(k+m)},$$

as desired. □

Generally, knowing that a (verbal) subgroup is closed can be helpful when working with profinite groups. That is why we should finish this introduction by pointing out the following critical result due to Jaikin-Zapirain.

Theorem 4.11 (cf. [44, Theorem 1.1])**.** *Let $w \in F_k$ be a word in k variables. Then w has finite width in all finitely generated pro-p groups if and only if $w \notin \delta_2(F_k)\,(F_k')^p$.*

This chapter is devoted to proving that compact R-analytic groups are verbally concise. It is worth mentioning that when $\operatorname{char} R = 0$, this result is a direct consequence of Theorem 2.27 –every compact R-analytic group is linear– together with Merzjalkov's Theorem –linear groups are verbally concise–. But in spite of that, it is interesting to provide an independent proof, which is valid for any pro-p domain regardless of the characteristic.

Furthermore, this general result provides yet another evidence for a positive answer to the question of whether compact analytic groups are linear in positive characteristic.

4.1 Conciseness in R-standard groups

In the class of R-standard groups, conciseness is straightforward:

Proposition 4.12. *Let S be an R-standard group and let w be a word such that $w\{S\}$ is finite. Then $w(S) = \{1\}$.*

Proof. Firstly, S can be identified with $\left(\mathfrak{m}^N\right)^{(d)}$, where N is the level and d the dimension of S, such that the multiplication and the inversion are defined by two tuples of power series and the identity is $\mathbf{0}$. Consequently, the word map w is a single power series $\mathbf{W} \in R[[X_1, \ldots, X_{dk}]]^{(d)}$, where k is the number of variables of the word. Since $w\{S\}$ is finite and the word map is continuous, $\mathbf{W}$ is locally constant, so by Lemma 1.8 $\mathbf{W}$ is constant, i.e $\mathbf{W}(X_1, \ldots, X_{dk}) = \mathbf{W}(\mathbf{0}, \ldots, \mathbf{0}) = \mathbf{0}$, and thus $w\{S\} = \{\mathbf{0}\}$. □

We should take notice of a couple of consequences of the previous result. On the one hand, any R-analytic group G satisfies a weaker version of the conciseness conjecture:

Corollary 4.13. *Let G be an R-analytic group and let w be a word such that $w\{G\}$ is finite. There exists an open R-standard subgroup S where w is a law, that is, $w(S) = \{1\}$.*

Proof. According Lemma 1.21, there exists an open R-standard subgroup S of G. Since $|w\{S\}| \leq |w\{G\}|$, from Proposition 4.12 it follows that $w(S) = \{1\}$. □

On the other hand, if G is a compact R-analytic group such that $w\{G\}$ is finite, we can obtain the set of w-values just by looking at a transversal of a convenient R-standard subgroup. Indeed, let G be a compact R-analytic group, $S \trianglelefteq_o G$ an open normal R-standard subgroup whose conjugation maps are strictly analytic, which exists by Lemma 1.23 (provided that R is not a PID), and let T be a left transversal for S in G. We should bring the atlas induced by S to mind, namely the atlas $\{(tS, \phi_t)\}_{t\in T}$ where $\phi_t(x) = \phi(t^{-1}x)$. As a consequence of Lemma 1.24, the R-analytic word map $w\colon G^{(k)} \to G$, which is nothing but an adequate composition of multiplication and inversion maps, is given by a single tuple of power series on the open subset $t_1S \times \cdots \times t_kS$ ($t_i \in T$), i.e. there exists a tuple of power series $\mathbf{W}_{t_1,\ldots,t_k} \in R[[X_1, \ldots, X_{dk}]]^{(d)}$ such that

$$\phi_p(w(x_1, \ldots, x_k)) = \mathbf{W}_{t_1,\ldots,t_k}(\phi_{t_1}(x_1), \ldots, \phi_{t_k}(x_k)) \;\; \forall x_j \in t_jS, \tag{4.2}$$

where p is the element of T such that $w(t_1, \ldots, t_k)p^{-1} \in S$.

If we further assume that $w\{G\}$ is finite, the continuous map w is locally constant, so by Lemma 1.8, $\mathbf{W}_{t_1,\ldots,t_k}$ is constant, i.e.

$$\mathbf{W}_{t_1,\ldots,t_k}(X_1, \ldots, X_{dk}) = \mathbf{c} \in R^{(d)}.$$

That is, $\phi_p(w(x_1, \ldots, x_k)) = \mathbf{c}$ for all $x_j \in t_jS$. In other words,

Proposition 4.14. *Let G be a compact R-analytic group, let w be a word and let S be an open normal R-standard subgroup whose conjugation maps are strictly analytic. If $w\{G\}$ is finite, then S is marginal for w.*

4.2 Conciseness in compact $\mathbb{F}_p[[t]]$-analytic groups

The demonstration technique will consist in reducing the problem to a group that is analytic over a pro-p domain of Krull dimension one, by using the change of rings described in Section 2.2. Hence, firstly we shall deal with the 1-dimensional case. Compact p-adic analytic groups are verbally concise, as they are linear by virtue of Corollary 2.6. Hence, in view of Corollary 1.44 we can restrict to the case $R = \mathbb{F}_p[[t]]$.

We shall proceed via several technical results. Some of them will be proved, whilst others will be simply stated. We start with the following "folklore" result:

Lemma 4.15. *Let G be a group and let w be a word such that $w\{G\}$ is finite. Then $w(G)'$ is finite, and $w(G)$ is finite if and only every w-value in G has finite order.*

Proof. Let $g \in G$. By Lemma 4.3, $w\{G\}^g \subseteq w\{G\}$, i.e. for all $x \in w\{G\}$ the conjugacy class x^G is contained in $w\{G\}$. Hence

$$|G : C_G(x)| = \left|w^G\right| \leq |x\{G\}|,$$

so $C_G(x)$ has finite index in G. Therefore, $C_G(w(G)) = \cap_{x\in w\{G\}} C_G(x)$ has finite index in G, and so $|w(G) : Z(w(G))|$ is also finite. Thus, by Schur's Theorem (cf. [65, Theorem 10.1.4]), $w(G)'$ is finite.

Finally, if $w(G)$ is finite, it must have finite exponent. Conversely, suppose that the elements of $w\{G\}$ have finite order, then $w(G)/w(G)'$ is an abelian group finitely generated by elements of finite order, in particular, it is finite; and the finiteness of $w(G)'$ yields the result. □

Moreover, we will need the following Schur-type result:

Lemma 4.16 (cf. [45, Proposition 5.1])**.** *Let G be a group with a nilpotent normal subgroup N. Suppose that $\frac{NZ(G)}{Z(G)}$ has finite exponent. Then $[N, G]$ has finite exponent.*

Proof. The Hall-Petrescu formula (see [38, III.9.4]) states that for all $m \in \mathbb{N}$,

$$x^m y^m = (xy)^m c_2(x, y)^{\binom{m}{2}} \dots c_m(x, y)^{\binom{m}{m}}, \tag{4.3}$$

where $c_r(x, y) \in \gamma_r(\langle x, y\rangle)$.

Let m be the exponent of $\frac{NZ(G)}{Z(G)}$, according to (4.3), for all $n \in N$ and $g \in G$:

$$[n, g]^m \equiv n^{-m}(n[n, g])^m = n^{-m}(n^g)^m = [n^m, g] = 1 \mod \gamma_2(K), \tag{4.4}$$

where $K := \langle n, [n, g]\rangle \leq N$. Moreover, by (4.3) and (4.4), for every $l \in \mathbb{N}$ we have

$$([n_1, g_1] \dots [n_l, g_l])^m \equiv [n_1, g_1]^m \dots [n_l, g_l]^m \equiv 1 \mod \gamma_2(N).$$

Let $\eta(m)$ be[†] the order of the largest 2-generated nilpotent group of exponent m. It suffices to prove that $\gamma_2(N)^{\eta(m)} = \{1\}$. For that, let $x, y \in N$ and $H = \langle x, y\rangle \leq N$. Since $H/Z(H)$ is a 2-generated nilpotent group of exponent dividing

[†] This integer exists because, as Baer [4] proved, nilpotent groups satisfy the Burnside problem (see [20, Theorem 2.23]).

m, it is finite and $k = |H : Z(H)|$ divides $\eta(m)$. Moreover, $\theta\colon H \to Z(H)$, $h \mapsto h^k$ is the transfer of H into $Z(H)$ (compare with the proof of [65, Theorem 10.1.3]). In particular, θ is a homomorphism, so $(xy)^k = x^k y^k$. Since x^k and y^k commute and k divides $\eta(m)$, we have that

$$(xy)^{\eta(m)} = x^{\eta(m)} y^{\eta(m)} \ \forall x, y \in N.$$

In particular, $\theta'\colon N \to Z(N)$, $n \mapsto n^{\eta(m)}$, is a homomorphism, and since $\operatorname{im}\theta'$ is abelian, $\gamma_2(N)$ must be contained in $\ker\theta'$, that is, $\gamma_2(N)^{\eta(m)} = \{1\}$. □

The upcoming auxiliary results use ideas from the theory of linear algebraic groups. The reader is directed to [37] for a thorough account of the theory behind these results.

Let K be an algebraically closed field. For our purposes a *linear algebraic group* will be a closed subgroup $\mathcal{G}$ of $\mathrm{GL}_n(K)$ endowed with the Zariski topology, and the *identity component* of $\mathcal{G}$ is the connected component of the identity.

Proposition 4.17 (cf. [37, Proposition 7.3])**.** *Let $\mathcal{G}$ be a linear algebraic group.*

(*i*) *The identity component of $\mathcal{G}$ is a normal subgroup of finite index.*

(ii) *Let $\mathcal{H} \leq \mathcal{G}$ be a closed connected subgroup of finite index, then $\mathcal{H} = \mathcal{G}^\circ$.*

The identity component of a linear algebraic group $\mathcal{G}$ is the unique normal closed connected subgroup of finite index in $\mathcal{G}$.

Furthermore, a matrix is *unipotent* if its unique eigenvalue is 1, and a subgroup of $\mathrm{GL}_n(K)$ is a *unipotent subgroup* if all its elements are unipotent. The key structural result about unipotent groups is that any unipotent subgroup is a conjugate of a subgroup of $U_n(K)$, the group of upper triangular matrices with 1's along the diagonal (see [37, Corollary 17.5]). In particular, every unipotent group G is nilpotent, and when the base field is of positive characteristic, G has finite exponent, compare with [38, Chapter III, Lemma 16.2 and Theorem 16.5] (although in this reference the base field is finite, the arguments are still valid for fields of positive characteristic).

Additionally, given an arbitrary linear algebraic group $\mathcal{G}$, the *unipotent radical* of $\mathcal{G}$, denoted $R_u(\mathcal{G})$, is the subgroup consisting of all the unipotent elements of $\mathcal{G}$, and it is also characterised as the largest connected unipotent subgroup of $\mathcal{G}$. Since $R_u(\mathcal{G})$ is connected and nilpotent, it is contained in the *soluble radical* of $\mathcal{G}$, namely the identity component of the largest soluble subgroup of $\mathcal{G}$. A non-trivial linear algebraic group $\mathcal{G}$ is said to be *reductive*, when it is connected and $R_u(\mathcal{G})$ is trivial.

Although all those constructions play an important rôle in the theory of algebraic groups, we have just summarised the definitions and the relations between them, considering that we will simply use the following technical result:

Proposition 4.18 (cf. [37, Lemma 17.9])**.** *Let $\mathcal{G}$ be a connected linear algebraic group and let $\mathcal{N}$ be its soluble radical. Then $[\mathcal{N},\mathcal{G}]$ is unipotent.*

Proof. Let $\mathcal{R}_u$ be the unipotent radical of $\mathcal{G}$. Then $\mathcal{G}/\mathcal{R}_u$ is reductive, so according to [37, Lemma 17.9], $\mathcal{N}/\mathcal{R}_u \subseteq Z\left(\mathcal{G}/\mathcal{R}_u\right)$, and therefore $[\mathcal{N},\mathcal{G}] \subseteq \mathcal{R}_u$. □

Now, we can prove the desired result:

Theorem 4.19. *Compact $\mathbb{F}_p[[t]]$-analytic groups are verbally concise.*

Proof. Let G be a compact $\mathbb{F}_p[[t]]$-analytic group and let w be a word such that $w\{G\}$ is finite. Firstly, by Lemma 4.15, $w(G)'$ is finite, and thus up to a quotient we can assume that $w(G)$ is a finitely generated abelian subgroup.

According to Corollary 4.13, there exists an open $\mathbb{F}_p[[t]]$-standard group S where w is a law. For abbreviation, Z stands for $Z(S)$ and K for the algebraic closure of the local field $\mathbb{F}_p((t))$. By Proposition 2.3, S/Z is a linear group over $\mathbb{F}_p[[t]]$, so it is also linear over the fields $\mathbb{F}_p((t))$ and K. According to the topological Tits alternative (loc. cit.) S/Z contains either an open soluble subgroup or a dense free subgroup. But S/Z satisfies an identity, so it must be virtually soluble.

Let $\mathcal{S}$ be the Zariski closure of S/Z in $\mathrm{GL}_n(K)$, which is also virtually soluble by Proposition 3.34. Let $\mathcal{N}$ be the largest normal soluble subgroup of $\mathcal{S}$ and $\mathcal{N}^\circ$ its identity component, i.e. the soluble radical of the algebraic group $\mathcal{S}$. In view of Proposition 4.17(i), $\mathcal{N}^\circ$ has finite index in $\mathcal{S}$, so according to Proposition 4.17(ii), $\mathcal{N}^\circ = \mathcal{S}^\circ$. Let N/Z be the intersection of S/Z with $\mathcal{N}^\circ$, then, passing to the normal core if necessary, we can assume that N is a normal subgroup of finite index in G.

Besides, according to Proposition 4.18, $[\mathcal{N}^\circ,\mathcal{N}^\circ]$ is unipotent. In particular, $[\mathcal{N}^\circ,\mathcal{N}^\circ]$ is nilpotent and, since K has characteristic p, it has finite exponent. Therefore, $[N,N]Z/Z$ is nilpotent of finite exponent. Thus $[N,N]Z$ is nilpotent and according to Lemma 4.16, $H := [N,N,S]$ has finite exponent.

On the one hand,

$$H = [N,N,S] \geq [N,N,N],$$

and so G/H is virtually nilpotent of class at most 2. Since $|w\{G/H\}| \leq |w\{G\}|$ and G/H is virtually nilpotent, we conclude that $w\left(G/H\right)$ is finite by Turner-Smith's Theorem (cf. [72, Corollary 2]).

On the other hand, $w(G) \cap H$ is a finitely generated abelian group of finite exponent, so it is finite. Finally, the isomorphism

$$w\left(\frac{G}{H}\right) = \frac{w(G)H}{H} \cong \frac{w(G)}{w(G) \cap H}$$

yields the result. □

Corollary 4.20. *Let R be a pro-p domain of Krull dimension one. Every compact R-analytic group is verbally concise.*

4.3 Conciseness in compact R-analytic groups

Now, we are primed to prove the principal result.

Theorem 4.21. *Every compact R-analytic group is verbally concise.*

Proof. Suppose, in view of Corollary 4.20, that R has Krull dimension at least 2. Let G be a compact R-analytic group and let w be a word in k variables such that $w\{G\}$ is finite. By virtue of Lemma 4.15, it suffices to prove that every w-value is of finite order.

By Lemma 1.23, there exists an open normal R-standard subgroup (S, ϕ) such that for every $g \in G$ the conjugation map $c_g \colon S \to S$, $x \mapsto x^g$ is strictly analytic. If $n = |G : S|$, then $w^n\{G\} \subseteq S$ and, by Lemma 4.15, $w^n(G)$ is finite if and only if $w(G)$ is finite. Therefore, without loss of generality assume that $w\{G\} \subseteq S$.

Let $(P, \mathfrak{m})$ be the principal ideal pro-p domain $\mathbb{Z}_p$ if $\operatorname{char} R = 0$ or $\mathbb{F}_p[[t]]$ if $\operatorname{char} R = p$. According to Cohen's Structure Theorem (see Theorem 1.2), R is a finitely generated integral extension of $P[[t_1, \ldots, t_m]]$, where $m = \dim_{\mathrm{Krull}}(R) - 1$. For each $a \in \mathfrak{m}^{(m)}$, let s_a be the evaluation epimorphism $s_a \colon P[[t_1, \ldots, t_m]] \to P$, $F(t_1, \ldots, t_m) \mapsto F(a)$. By Corollary 2.13, s_a extends to a continuous ring epimorphism $\tilde{s}_a \colon R \to Q$, where, in view of Remark 2.14, $Q = (Q, \mathfrak{n})$ is a pro-p domain and a finitely generated integral extension of P, in particular, Q has Krull dimension 1.

Fix $a \in \mathfrak{m}^{(m)}$, throughout this proof we use $\mathbf{W}_a$ to denote $\mathbf{W}_{\tilde{s}_a}$ for any tuple of power series $\mathbf{W}$ (see Section 2.2). In particular, if $\mathbf{F}$ is the formal group law of S, $\mathbf{F}_a$ is the formal group law $\mathbf{F}_{\tilde{s}_a}$ (see Corollary 2.8). Let T be a left transversal for S in G, and assume that $1 \in T$. We will use the atlas induced by S, namely $\{(tS, \phi_t)\}_{t \in T}$ where $\phi_t(x) := \phi\left(t^{-1}x\right)$ (compare with Section 1.3).

Using Lemma 2.10, we define the Q-standard group $L := \left(\mathfrak{n}^N\right)^{(d)}$, whose group operation is given by $\mathbf{F}_a$, and a compact Q-analytic group $H := T \times L$, that can be regarded as an overgroup of L and whose group operation, say $*_a$, is defined as in (2.3). Recall that the Q-analytic structure of H is given by the atlas $\{(tL, \psi_t)\}_{t\in T}$ where $\psi_t(t,l) = l$.

For the rest of the proof fix $(t_1, \dots, t_k) \in T^{(k)}$ and assume, by (4.2), that for any $\ell \in \mathbb{N}$, the word map w^ℓ is given in $t_1 S \times \cdots \times t_k S$ by the tuple of power series $\mathbf{W}^\ell$, that is, recalling that $w\{G\} \subseteq S$ we have that

$$\phi\left(w^\ell(x_1, \dots, x_k)\right) = \mathbf{W}^\ell\left(\phi_{t_1}(x_1), \dots, \phi_{t_k}(x_k)\right) \ \forall x_j \in t_j S$$

(even though in order to lighten the notation it is not written explicitly, the power series $\mathbf{W}^\ell$ also depends on $t_1, \dots, t_k$).

Let $w^\ell \colon H^{(k)} \to H$ be the word map w^ℓ with respect to the operation $*_a$ of H. By Lemma 2.7 and Remark 2.11,

$$\psi_1\left(w^\ell(x_1, \dots, x_k)\right) = \mathbf{W}_a^\ell\left(\psi_{t_1}(x_1), \dots, \psi_{t_k}(x_k)\right) \ \forall x_j \in t_j L.$$

Furthermore, according to Proposition 4.14, since $w\{G\}$ is finite, S is marginal for w. That is, w is constant in each open subset $t_1 S \times \cdots \times t_k S$. Hence, the word map $w \colon H^{(k)} \to H$ is constant in each $t_1 L \times \cdots \times t_k L$, and thus $|w\{H\}| \leq |H : L|^k = n^k$ is finite. According to Corollary 4.20, H is verbally concise, so there exists $\ell_a \in \mathbb{N}$ such that $w^{\ell_a}(H) = \{(1, \mathbf{0})\}$. In particular, by Lemma 1.8

$$\mathbf{W}_a^{\ell_a}(X_1, \dots, X_{dk}) = \mathbf{0}. \tag{4.5}$$

Define the following partition of the space $\mathfrak{m}^{(m)}$:

$$\mathfrak{m}_\ell = \left\{a \in \mathfrak{m}^{(m)} \middle| \mathbf{W}_a^\ell = \mathbf{0}\right\}, \ \ell \in \mathbb{N}.$$

Since $\mathfrak{m}^{(m)} = \bigcup_{\ell\in\mathbb{N}} \overline{\mathfrak{m}_\ell}$ and $\mathfrak{m}^{(m)}$ is complete, by the Baire Category Theorem there exists ℓ' such that $\overline{\mathfrak{m}_{\ell'}}$ contains a non-empty open subset $V \subseteq_o \mathfrak{m}^{(m)}$. Thus, $\mathbf{W}^{\ell'}$ is a constant tuple of power series, i.e.

$$\mathbf{W}^{\ell'}(X_1, \dots, X_{dk}) = (c_1, \dots, c_d) \in R^{(d)},$$

that whenever $a \in \mathfrak{m}_{\ell'}$, satisfies

$$(\tilde{s}_a(c_1), \dots, \tilde{s}_a(c_d)) = \mathbf{W}_a^{\ell'}(X_1, \dots, X_{dk}) = \mathbf{0}.$$

Consequently, $\tilde{s_a}(c_i) = 0$ for all $a \in \mathfrak{m}_{\ell'} \cap V$. Besides, $\mathfrak{m}_{\ell'} \cap V$ is dense in V, so by Corollary 2.15, we have that $c_i = 0$ for all $i \in \{1, \dots, d\}$. Finally, repeating the process for all the tuples in $T^{(k)}$, we obtain an integer ℓ such that $\phi\left(w^\ell(x_1, \dots, x_k)\right) = \mathbf{0}$ for all $x_i \in G$, and thus $w^\ell(G) = \{1\}$. □

It is known that uniformly powerful groups are torsion-free (cf. [24, Theorem 4.5]). Therefore, whenever H is a compact p-adic analytic group with open $\mathbb{Z}_p$-standard, and so uniformly powerful, subgroup $L \leq H$ and w is a word such that $w\{H\}$ is finite, then $w(H)$ is finite, and therefore $w^{|H:L|}\{H\}$ consists of finite order elements of the torsion-free group L. That is, $w^{|H:L|}$ is a law in H.

Using this fact, the preceding proof can be slightly simplified when R is a pro-p domain of characteristic zero. Indeed, in (4.5) we would have that $\mathbf{W}_a^{|G:S|} = \mathbf{0}$, independently of the evaluation $\tilde{s}_a$, and thus we can avoid the use of the Baire Category Theorem. Note that, in passing, it would prove that every w-value has order dividing $|G : S|$.

4.4 Strong conciseness in R-analytic groups

In the setting of profinite groups the concept of conciseness can be strengthened to that of strong conciseness (compare with [22] and [23]). More specifically, the word w is said to be *strongly concise* in a class of profinite groups $\mathcal{C}$, if for any $G \in \mathcal{C}$ such that $|w\{G\}| < 2^{\aleph_0}$, then $w(G)$ is finite.

Nonetheless, for compact R-analytic groups strong conciseness is equivalent to conciseness.

Lemma 4.22 (cf. [22, Proposition 2.1])**.** *Let X be a compact topological space and let Y be a* profinite space*, i.e. a compact Hausdorff totally disconnected topological space. Suppose that the continuous map $F \colon X \to Y$ is* nowhere constant*, that is, $F|_U$ is not constant for every open $U \subseteq_o X$. Then $|\operatorname{im} F| \geq 2^{\aleph_0}$.*

Proof. Let $U \subseteq_o X$ be a non-empty clopen (closed and open) subset (e.g. $U = X$). Since $F|_U$ is not constant, there exist two non-empty clopen subsets $U_1, U_2 \subseteq U$ such that $F(U_1) \cap F(U_2) = \varnothing$. Repeating this procedure, for each sequence $\mathbf{x} = (x_n)_{n\geq 1} \in \{1,2\}^{\mathbb{N}}$ we obtain a descending sequence of closed subsets

$$U \supseteq U_{x_1} \supseteq U_{x_1,x_2} \supseteq \cdots \supseteq U_{x_1,\ldots,x_n} \supseteq \ldots$$

Define $U_{\mathbf{x}} := \bigcap_{n\in\mathbb{N}} U_{x_1,\ldots,x_n}$, which is non-empty as X is compact. Accordingly, for each sequence $\mathbf{x} \in \{1,2\}^{\mathbb{N}}$, choose $u_{\mathbf{x}} \in U_{\mathbf{x}}$. Since $F(u_{\mathbf{x}}) \neq F(u_{\mathbf{y}})$, the set

$$\mathcal{S} := \left\{u_{\mathbf{x}} \mid \mathbf{x} \in \{1,2\}^{\mathbb{N}}\right\}$$

maps injectively to Y, and therefore $|F(X)| \geq |F(\mathcal{S})| = 2^{\aleph_0}$. □

Proposition 4.23. *Let M and N be compact R-analytic manifolds and $F\colon M \to N$ an R-analytic map such that $|\operatorname{im} F| < 2^{\aleph_0}$. Then $\operatorname{im} F$ is finite.*

Proof. Let $m = \dim M$ and $n = \dim N$. Since F is analytic, for each $x \in M$ there exists a regular R-chart (U_x, ϕ) of x in M, an R-chart (V_x, ψ) of $F(x)$ in N and a tuple of power series $\mathbf{G} \in R[[X_1, \ldots, X_m]]^{(n)}$ such that $F(U_x) \subseteq V_x$ and

$$\psi \circ F \circ \phi^{-1}(y) = \mathbf{G}(y) \ \forall y \in \phi(U_x).$$

Since $\phi(U_x) = z + \left(\mathfrak{m}^L\right)^{(m)}$, for some $z \in R^{(m)}$ and $L \in \mathbb{N}$, then U_x is a profinite space. Hence, $F|_{U_x}\colon U_x \to N$ is a continuous map between profinite spaces such that $|F(U_x)| < 2^{\aleph_0}$, so, by Lemma 4.22, there exists $V \subseteq_o U_x$ such that $F|_V$ is constant, and by Lemma 1.8, $F|_{U_x}$ is constant. By compactness, $M = \bigcup_{z \in Z} U_z$ for some finite subset $Z \subseteq M$, and particularly $|\operatorname{im} F| \leq |Z|$. □

In particular, whenever $|w\{G\}| < 2^{\aleph_0}$ in a compact R-analytic group G, then $w\{G\}$ is finite. Therefore, the main result can be restated as follows:

Corollary 4.24. *Every word is strongly concise in the class of compact R-analytic groups.*

Furthermore, as a byproduct we obtain a lower bound for the cardinality of the set of word-values in some finitely generated R-analytic groups. More specifically, in [45, Theorem 1.3], the authors established a criterion to isolate finitely generated compact R-analytic groups satisfying a law, proving that they are p-adic analytic.

Corollary 4.25. *Let R be a pro-p domain of characteristic p or Krull dimension at least 2, and let G be a non-discrete finitely generated R-analytic group. For every word w, we have that $|w\{G\}| \geq 2^{\aleph_0}$.*

Proof. According to Lemma 1.21, there exists an open R-standard subgroup S of G. Suppose that $|w\{G\}| < 2^{\aleph_0}$, then $|w\{S\}| < 2^{\aleph_0}$. Since S is compact, by Proposition 4.12 and Proposition 4.23, w is a law in S so, by [45, Theorem 1.3], S admits both a p-adic analytic and an R-analytic structure, which is a contradiction with Theorem 1.45(i). □

4.5 Notes

Supplementary material regarding words and verbal width can be found in the detailed surveys of Neumann [61] and Segal [67]. Moreover, the original papers of P. Hall [33] and Turner Smith [72] are worth reading, as basic notions are established there. In addition, the concept of strong conciseness in profinite groups is introduced and extensively studied in the articles [22] and [23]. Finally, we have adopted the term “robust word” from [69].

The primary outcome of the chapter is Theorem 4.21, published in [76]. However, Theorem 4.19 is an important stepping stone to it, and the work [45] of Jaikin-Zapirain and Klopsch has been a great inspiration to prove the latter.

A
Ado's Theorem over principal ideal domains

In Section 2.1, we called for a variant of Ado's Theorem. Indeed, we wanted the Lie algebra to be defined over a principal ideal domain, not necessarily a field; and at the same time, we also demanded the degree of the matricial representation to depend only on the rank of the Lie algebra as a free R-module. There are well-established references for the former –it is proved in [19] and [74]–, but we were unable to find a precise reference for the latter. For that reason, for the sake of this thesis's completeness, this appendix is devoted to proving in detail the required version of the theorem, by collecting facts and ideas that appear in the mathematical literature concerning this problem.

A.1 Introduction

Throughout this appendix R will be a principal ideal domain (PID) of characteristic zero and $K = \mathrm{Frac}(R)$ its field of fractions. Moreover, unless it is explicitly specified, an R-algebra need not be associative, commutative or with identity.

Recall that an *R-Lie algebra* is an R-module $\mathfrak{L}$ endowed with a *Lie bracket*, namely an antisymmetric bilinear map $[\cdot,\cdot]\colon \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ that satisfies *Jacobi's* *identity*, i.e.

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 \ \forall x, y, z \in \mathfrak{L}.$$

R-Lie algebra homomorphisms are linear maps that preserve Lie brackets, that is, a linear map $\phi\colon (\mathfrak{L}, [\cdot,\cdot]_{\mathfrak{L}}) \to (\mathfrak{L}', [\cdot,\cdot]_{\mathfrak{L}'})$ between R-Lie algebras such that

$$\phi\left([x,y]_{\mathfrak{L}}\right) = [\phi(x), \phi(y)]_{\mathfrak{L}'} \ \ \forall x, y \in \mathfrak{L}.$$

Examples of those algebras are *concrete Lie algebras*, namely algebras $\mathrm{End}_R(W)$ consisting of R-module endomorphisms over a free R-module W with the bracket operation given by

$$[f,g] = f \circ g - g \circ f.$$

Definition A.1. Let $\mathfrak{L}$ be an R-Lie algebra. A *representation* of $\mathfrak{L}$ is an R-Lie algebra homomorphism

$$\phi\colon \mathfrak{L} \to \mathrm{End}_R(W),$$

for some free R-module W. Moreover, ϕ is a *faithful representation* when it is an injective map.

The *degree* of the representation ϕ, denoted as $\deg\phi$, is the rank of W, and ϕ is said to be a *representation of finite rank* or a *finite representation* when $\deg\phi$ is finite. In this situation, $\mathrm{End}_R\,(W)$ and $\mathrm{M}_{\mathrm{rk}\,W}(R)$ (endowed with the Lie bracket $[A,B] = AB - BA$) are isomorphic R-Lie algebras, and consequently finite faithful representations are also referred to as *matricial representations*.

Lie algebras have been extensively studied over fields, and Ado's Theorem gives account of their representability:

Theorem A.2 (Ado [2])**.** *Suppose that R is a field of characteristic zero. Every finite dimensional R-Lie algebra $\mathfrak{L}$ has a finite faithful representation.*

More generally, if we require $\mathfrak{L}$ to be a free R-module, the preceding theorem can be generalised to PIDs:

Theorem A.3 (Churkin [19]; Weigel [74])**.** *Let R be a PID of characteristic zero and let $\mathfrak{L}$ be an R-Lie algebra which is a free R-module of finite rank. Then $\mathfrak{L}$ has a finite faithful representation.*

Henceforward, following the terminology in [74], an *R-Lie lattice* is an R-Lie algebra which is a finitely generated free R-module as well.

There are several proofs of Ado's Theorem (see, for instance, [41, Chapter VI, Section 2]), and from most of them we can conclude that the degree of the representation depends only on the vector space dimension of the Lie algebra. More precisely, let

$$\deg \mathfrak{L} := \min\{\deg \phi \mid \phi \text{ is a faithful representation of } \mathfrak{L}\}.$$

If R is a field of characteristic zero and $\mathfrak{L}$ is an R-Lie algebra of dimension r, then $\deg \mathfrak{L} \leq f(r)$ for some non-decreasing function $f \colon \mathbb{N}_0 \to \mathbb{N}$. For instance, from [13] and [60], we know that

$$\deg \mathfrak{L} \leq r + \alpha \frac{2^r}{\sqrt{r}}, \tag{A.1}$$

for some $\alpha \approx 2.763$ (we refer to [58, Section 1.1.2] for a detailed proof). There are various other works studying $\deg \mathfrak{L}$ over fields, and we ought to mention [8], [31] and [63].

For general PIDs, meanwhile, we can not directly make the same deduction from the initially-mentioned two references. In fact, in [74, Proposition 3.4], the finiteness of the degree-to-be follows from the fact that since R is Noetherian any ascending chain of ideals must be stationary, although we can not determine the number of ideals in the chain.

In view of this, we shall present a quantitative way of constructing a faithful representation of an R-Lie lattice. This procedure will be grounded on the ideas that appear in [8] and [63].

Theorem A.4. *Let $\mathfrak{L}$ be an R-Lie lattice of rank r. Then*

$$\deg \mathfrak{L} \leq r + \sqrt{\frac{r+1}{r}} 4^r.$$

Lastly, it is worth mentioning that the positive characteristic counterpart of Ado's Theorem is also true, as proved by Iwasawa [40]. In fact, the general version of Theorem A.2, without any restriction on the characteristic of the base field, is referred to as the Ado-Iwasawa Theorem. Nevertheless, in positive characteristic the result can be stated with much more generality:

Theorem A.5 (cf. [19, Theorem 3])**.** *Let R a ring of positive characteristic and $\mathfrak{L}$ an R-Lie lattice. Then there exist a free R-module W of finite rank and an injective R-Lie algebra homomorphism $\phi \colon \mathfrak{L} \hookrightarrow \operatorname{End}_R(W)$.*

In order to prove this theorem it suffices to reproduce the original proof word by word, and thus we obtain for $\deg \mathfrak{L}$ the same bound we already knew over fields, namely

$$\deg \mathfrak{L} \leq n^{\operatorname{rk}^3 \mathfrak{L}},$$

where $n = \operatorname{char} R$ (compare with [5, Section 6.2.4]).

Remarks. Thoughout the proofs we will use some basic properties of free R-modules over PIDs. Here is what we should take into account:

(i) submodules of a free R-module M are free, and they have rank at most $\operatorname{rk} M$.

Let M be an R-module and $N \leq M$ a submodule. The *isolator* of N in M is the submodule

$$\operatorname{Iso}_M(N) = \{x \in M \mid \exists\, r \in R \setminus \{0\} \text{ such that } rx \in N\},$$

and N is *isolated* in M if $\operatorname{Iso}_M(N) = N$.

(ii) $M/\operatorname{Iso}_M(N)$ is a torsion-free R-module.

(iii) If M is a free R-module, $N \leq M$ is an isolated submodule and M/N is finitely generated, then M/N is a free R-module and

$$\operatorname{rk}(M) = \operatorname{rk}(N) + \operatorname{rk}(M/N).$$

The integer $\operatorname{rk}(M/N)$ is referred to as the *corank* of N in M.

(iv) If M is a free R-module, $N \leq M$ is an isolated submodule and M/N is finitely generated, then N has a complementary in M, i.e. there exists a free R-module L such that $M = N \oplus L$.

A.2 Adjoint and regular representations

We shall introduce a couple of representations that arise naturally in every R-Lie lattice $\mathfrak{L}$. Firstly, $x \in \mathfrak{L}$ defines the linear endomorphism $\operatorname{ad}_x \colon \mathfrak{L} \to \mathfrak{L}$, $y \mapsto [x, y]$. This assignation gives the *adjoint representation* of degree $\operatorname{rk} \mathfrak{L}$, namely

$$\operatorname{Ad} \colon \mathfrak{L} \to \operatorname{End}_R(\mathfrak{L}),\ x \mapsto \operatorname{ad}_x,$$

which is an R-Lie algebra homomorphism in view of Jacobi's identity. Nonetheless, the adjoint representation is not faithful in general, as its kernel is the *centre* of the algebra,

$$Z(\mathfrak{L}) := \{x \in \mathfrak{L} \mid [x,y] = 0 \ \forall y \in \mathfrak{L}\}\,.$$

Therefore, if $\mathfrak{L}$ is a semisimple R-Lie algebra –it has no non-trivial abelian ideal–, the adjoint representation is faithful, and, in this situation, $\deg \mathfrak{L} \leq \operatorname{rk} \mathfrak{L}$.

In order to present the second representation, we must define the universal enveloping algebra:

Definition A.6 (cf. [41, Chapter V, Section 1])**.** Let $\mathfrak{L}$ be an R-Lie algebra. The *R-tensor algebra* of $\mathfrak{L}$ is

$$\mathbf{T}_R(\mathfrak{L}) = R \oplus \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \cdots \oplus \mathfrak{L}_i \oplus \ldots,$$

where $\mathfrak{L}_i := \mathfrak{L} \otimes \overset{(i)}{\ldots} \otimes \mathfrak{L}$. Each $\mathfrak{L}_i$ is an R-module with the natural R-module structure of tensorial modules and the multiplication in $\mathbf{T}_R(\mathfrak{L})$ is defined by the rule

$$(x_1 \otimes \cdots \otimes x_i) \otimes (y_1 \otimes \cdots \otimes y_j) = x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_j.$$

Let $\mathfrak{R}$ be the ideal of $\mathbf{T}_R(\mathfrak{L})$ generated by the elements

$$[x,y] - (x \otimes y - y \otimes x),\ \ x, y \in \mathfrak{L}.$$

The *universal enveloping algebra* of $\mathfrak{L}$ is the associative R-algebra with identity

$$\mathcal{U}_R(\mathfrak{L}) := \frac{\mathbf{T}_R(\mathfrak{L})}{\mathfrak{R}}.$$

Identifying $\mathfrak{L}$ with $\mathfrak{L}_1$, we obtain a homomorphism $\iota\colon \mathfrak{L} \to \mathcal{U}_R(\mathfrak{L})$, which, since R is a PID, is injective whenever $\mathfrak{L}$ is finitely generated (see [74, Theorem 3.2]). Hence, for simplicity we can assume that $\mathfrak{L} \subseteq \mathcal{U}_R(\mathfrak{L})$. Besides, in order to simplify the notation, the element $x_{i_1} \otimes \cdots \otimes x_{i_k}$ will be written as the monomial $x_{i_1} \ldots x_{i_k}$. The universal algebra is described by the Poincaré-Birkhoff-Witt Theorem:

Theorem A.7 (cf. [74, Theorem 3.2])**.** *Let $\mathfrak{L}$ be an R-Lie lattice with basis $\{x_1, \ldots, x_r\}$. Then $\mathcal{U}_R(\mathfrak{L})$ is a free R-module, and the monomials*

$$\{x_1^{\alpha_1} \ldots x_r^{\alpha_r} \mid \alpha_i \in \mathbb{N}_0\} \tag{A.2}$$

form a basis.

Indeed, given two monomials their product can be expressed as a linear combination of monomials of the form (A.2) by successively applying the identity $x_j x_i = x_i x_j - [x_i, x_j]$ to reorder the terms until all the involved monomials have the required order.

Any R-Lie lattice $\mathfrak{L}$ acts by left multiplication on $\mathcal{U}_R(\mathfrak{L})$. Indeed, for each $x \in \mathfrak{L}$ we get the R-linear endomorphism $\ell_x \colon \mathcal{U}_R(\mathfrak{L}) \to \mathcal{U}_R(\mathfrak{L})$, $u \mapsto xu$. On the one hand, for every $x, y \in \mathfrak{L}$

$$[x, y] = xy - yx$$

in $\mathcal{U}_R(\mathfrak{L})$, and therefore $\mathcal{L} \colon \mathfrak{L} \to \mathrm{End}_R(\mathcal{U}_R(\mathfrak{L}))$, $x \mapsto \ell_x$ is an R-Lie algebra homomorphism. On the other hand, since the universal enveloping algebra has an identity, whenever $x \neq y$ we have that

$$\ell_x(1) = x \neq y = \ell_y(1),$$

and so $\mathcal{L}$ is a faithful representation, called (left) *regular representation*.

However, since $\mathcal{U}_R(\mathfrak{L})$ is of infinite rank, the regular representation is not finite. Nevertheless, there is a property that characterises the universal enveloping algebra, and as a consequence of it any finite representation factors though $\mathcal{U}_R(\mathfrak{L})$.

Theorem A.8 (Universal property, cf. [11, Chapter I, § 2.1, Proposition 1])**.** *Let $\mathfrak{L}$ be an R-Lie lattice, A an associative R-algebra with identity together with the Lie bracket $[a, b] = ab - ba$ $(a, b \in A)$ and an R-Lie algebra homomorphism $\psi \colon \mathfrak{L} \to (A, [,])$. Then, there exists a unique R-algebra homomorphism $\psi^* \colon \mathcal{U}_R(\mathfrak{L}) \to A$ such that $\psi = \psi^* \circ \iota$. That is, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{U}_R(\mathfrak{L}) & & \\ {\scriptstyle \iota}\uparrow & \overset{\psi^*}{\searrow} & \\ \mathfrak{L} & \xrightarrow{\ \psi\ } & A. \end{array}$$

Actually, the name of $\mathcal{U}_R(\mathfrak{L})$ comes from this universal property. Further, it can be proved that any R-algebra that satisfies the universal property is isomorphic to $\mathcal{U}_R(\mathfrak{L})$ (compare with [41, Chapter V, Theorem 1.1]).

A.3 Ado's Theorem

In order to prove Ado's Theorem we will usually move from the R-Lie lattice $\mathfrak{L}$ to the K-vector space $\mathfrak{L}_K := \mathfrak{L} \otimes_R K$. Note that $\mathfrak{L}_K$ is a K-Lie algebra, whose Lie bracket is nothing but the K-linear application induced by the Lie bracket of $\mathfrak{L}$. We should bear in mind the following facts:

- $\mathfrak{L}_K$ is K-vector space of dimension $\operatorname{rk}\mathfrak{L}$,
- if $\mathfrak{L} = \langle x_1, \ldots, x_r \rangle_R$ then $\mathfrak{L}_K = \langle x_1, \ldots, x_r \rangle_K$, and
- whenever $\mathfrak{I} \trianglelefteq \mathfrak{L}_K$ is an ideal, then $\mathfrak{I} \cap \mathfrak{L} \trianglelefteq \mathfrak{L}$ is an isolated ideal.

We will prove the theorem in three steps:

A.3.1 Nilpotent Lie lattices

First of all, let us suppose that $\mathfrak{L}$ is a nilpotent R-Lie lattice of rank r. Recall that the *lower central series* of $\mathfrak{L}$ is defined as:

$$\gamma_1(\mathfrak{L}) := \mathfrak{L},\ \gamma_i(\mathfrak{L}) := [\gamma_{i-1}(\mathfrak{L}), \mathfrak{L}]\ \forall i \geq 2,$$

and that $\mathfrak{L}$ is nilpotent if there exists an integer $c \in \mathbb{N}$ such that $\gamma_{c+1}(\mathfrak{L}) = \{0\}$. The smallest of such integers, when it exists, is the *nilpotency class* of the R-Lie lattice. Considering that the tensor product is linear, we can easily prove the following:

Lemma A.9. *Let $\mathfrak{L}$ be an R-Lie lattice and let $\mathfrak{I}, \mathfrak{H} \trianglelefteq \mathfrak{L}$ be ideals. Then,*

$$[\mathfrak{I} \otimes_R K, \mathfrak{H} \otimes_R K] = [\mathfrak{I}, \mathfrak{H}] \otimes_R K.$$

Thus, if $\mathfrak{L}$ is a nilpotent R-Lie lattice of nilpotency class c, $\mathfrak{L}_K$ is a nilpotent K-Lie algebra of nilpotency class c.

Therefore, $\mathfrak{L}_K$ is nilpotent of nilpotency class say c, i.e

$$\mathfrak{L}_K = \gamma_1(\mathfrak{L}_K) > \cdots > \gamma_i(\mathfrak{L}_K) > \cdots > \gamma_{c+1}(\mathfrak{L}_K) = \{0\}$$

is a strictly descending chain of K-vector spaces and thus $c \leq \dim_K \mathfrak{L}_K = r$. Define now the isolated ideals $\mathfrak{L}_i := \gamma_i(\mathfrak{L}_K) \cap \mathfrak{L} \trianglelefteq \mathfrak{L}$, and choose a basis $\{x_1, \ldots, x_r\}$ of $\mathfrak{L}$ such that the first $x_1, \ldots, x_{r_1}$ elements are a basis for $\mathfrak{L}_c$, the first $x_1, \ldots, x_{r_2}$ ($r_2 > r_1$) elements form a basis for $\mathfrak{L}_{c-1}$ and so forth. According to Theorem A.7, the monomials

$$\mathbf{x}^\alpha := x_1^{\alpha_1} \ldots x_r^{\alpha_r},\ \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}_0^{(r)}$$

form a basis for the universal enveloping algebra $\mathcal{U}_R(\mathfrak{L})$, and accordingly we can define a *weight function* $\omega \colon \mathcal{U}_R(\mathfrak{L}) \to \mathbb{N}_0 \cup \{\infty\}$ in the following fashion:

$$\omega(x_i) = \max\{m \mid x_i \in \mathfrak{L}_m\}, \qquad \omega(\mathbf{x}^\alpha) = \textstyle\sum_{i=1}^r \alpha_i \omega(x_i),$$

$$\omega\left(\textstyle\sum_\alpha c_\alpha \mathbf{x}^\alpha\right) = \min\left\{\omega(\mathbf{x}^\alpha) \mid c_\alpha \neq 0\right\} \qquad \text{and } \omega(0) = \infty.$$

For each $m \in \mathbb{N}_0$, we define

$$\mathfrak{U}^m(\mathfrak{L}) := \{u \in \mathcal{U}_R(\mathfrak{L}) \mid \omega(u) > m\}$$

–when the lattice is clear from the context, we will simply write $\mathfrak{U}^m$–.

Let us show that $\mathfrak{U}^m(\mathfrak{L}) \trianglelefteq \mathcal{U}_R(\mathfrak{L})$ is an isolated ideal:

(i) Note that $0 \in \mathfrak{U}^m$ for all $m \in \mathbb{N}$ and that

$$\omega(rx) = \omega(x) \text{ and } \omega(x+y) \geq \min\left\{\omega(x), \omega(y)\right\},$$

for all $r \in R \setminus \{0\}$ and $x, y \in \mathcal{U}_R(\mathfrak{L})$. Additionally, since $\omega([x_i, x_j]) \geq \omega(x_i) + \omega(x_j)$ for every $i, j \in \{1, \ldots, r\}$, we have that

$$\omega(xy) \geq \omega(x) + \omega(y). \tag{A.3}$$

Consequently, $\mathfrak{U}^m$ is an ideal.

(ii) Let $r \in R \setminus \{0\}$,

$$rx \in \mathfrak{U}^m \implies \omega(x) = \omega(rx) > m \implies x \in \mathfrak{U}^m,$$

i.e. $\mathfrak{U}^m$ is isolated in $\mathcal{U}_R(\mathfrak{L})$.

Moreover, $\mathcal{U}_R(\mathfrak{L})/\mathfrak{U}^m$ is a finitely generated R-module, as it is generated by

$$\mathcal{B}_m = \left\{\mathbf{x}^\alpha + \mathfrak{U}^m \mid \omega(\mathbf{x}^\alpha) \leq m\right\}.$$

Therefore, $\mathfrak{U}^m$ is an isolated ideal of finite corank, and in view of (A.3), for every $x \in \mathfrak{L}$ we obtain that $\ell_x(\mathfrak{U}^m) \subseteq \mathfrak{U}^m$. In consequence, for any m the regular representation induces the finite representation

$$\mathcal{L}_m \colon \mathfrak{L} \to \operatorname{End}_R\left({}^{\mathcal{U}_R(\mathfrak{L})}/_{\mathfrak{U}^m}\right), \ x \mapsto \ell_x,$$

whose kernel is $\mathfrak{L} \cap \mathfrak{U}^m$ –with an abuse of notation, whenever $f \in \operatorname{End}_R(\mathcal{U}_R(\mathfrak{L}))$ satisfies $f(\mathfrak{X}) \subseteq \mathfrak{X}$ for some ideal $\mathfrak{X} \trianglelefteq \mathcal{U}_R(\mathfrak{L})$, we will still call f to the element in

$\mathrm{End}_R\left({}^{\mathcal{U}_R(\mathfrak{L})}/_{\mathfrak{X}}\right)$ that sends $x+\mathfrak{X}$ to $f(x)+\mathfrak{X}$–.

Besides, $\mathfrak{L}\cap\mathfrak{U}^c(\mathfrak{L})=\{0\}$. Indeed, whenever $x=\sum_{i=1}^r \alpha_i x_i\in\mathfrak{L}$, then

$$\omega(x)=\omega\left(\sum_{i=1}^r \alpha_i x_i\right)\le \max_{i=1,\ldots,r}\omega(x_i)=c.$$

Therefore, $\mathcal{L}_c$ is a finite faithful representation of $\mathfrak{L}$, and in order to bound its degree it suffices to determine an upper bound for $|\mathcal{B}_c|$. All the monomials in $\mathcal{B}_c$ have weight at most c, so they have polynomial degree at most c. Moreover, the number of monomials of polynomial degree at most c in r variables is exactly the number of monomials of polynomial degree c in $r+1$ variables (by adding an auxiliary variable, homogenise the monomials such that they have polynomial degree c).

Lemma A.10. *The number of monomials in r variables and of polynomial degree c is $\binom{r+c-1}{c}$.*

This observation gives us a simple bound (compare with [31, Corollary 5.1]):

$$\deg\mathfrak{L}\le \mathrm{rk}\left(\frac{\mathcal{U}_R(\mathfrak{L})}{\mathfrak{U}^c(\mathfrak{L})}\right)\le\binom{r+c}{c}.$$

We conclude this section by giving a not very sharp bound for $\deg\mathfrak{L}$ in terms of r. In fact, $c\in\{1,\ldots,r\}$, that is, $c=\alpha r$ where $\alpha\in\{{}^1/_r,\ldots,1\}$. Remember that according to the Stirling approximation formula,

$$\sqrt{2\pi r}\left({}^r/_e\right)^r\le r!\le\sqrt{2\pi r}\left({}^r/_e\right)^r e^{\frac{1}{12r}}\ \forall r\in\mathbb{N},$$

and therefore,

$$\begin{aligned}\binom{r+\alpha r}{\alpha r}&\le\frac{\sqrt{2\pi(r+\alpha r)}(r+\alpha r)^{r+\alpha r}}{\sqrt{2\pi\alpha r}(\alpha r)^{\alpha r}\sqrt{2\pi r}\ r^r}e^{\frac{1}{12(1+\alpha)r}}\\&\le\frac{e^{1/12(1+\alpha)r}}{\sqrt{2\pi r}}\frac{\sqrt{1+\alpha}}{\sqrt{\alpha}}\left(\frac{(1+\alpha)^{1+\alpha}}{\alpha^\alpha}\right)^r.\end{aligned}$$

In addition, the left-hand side and the right-hand side terms are asymtotically equivalent as r tends to infinity. Further, since $\alpha\in\left[{}^1/_r,1\right]$, then

$$\frac{\sqrt{1+\alpha}}{\sqrt{\alpha}}\le\sqrt{r+1}\ \text{and}\ \frac{(1+\alpha)^{1+\alpha}}{\alpha^\alpha}\le 4.$$

That is,

$$\frac{e^{1/12(1+\alpha)r}}{\sqrt{2\pi}}\frac{1}{\sqrt{r}}\frac{\sqrt{1+\alpha}}{\sqrt{\alpha}}\left(\frac{(1+\alpha)^{1+\alpha}}{\alpha^{\alpha}}\right)^{r} \leq \sqrt{\frac{r+1}{r}}4^{r}.$$

Consequently,

$$\binom{r+c}{c} \leq \sqrt{\frac{r+1}{r}}4^{r} \quad \forall c \in \{1,\ldots,r\}. \tag{A.4}$$

A.3.2 Splittable R-Lie lattices

The second step consists on obtaining a suitable R-Lie algebra representation for the so-called splittable R-Lie lattices.

Since the sum of nilpotent ideals is again nilpotent (compare with [41, Chapter I, Proposition 7.6]), every R-Lie lattice $\mathfrak{L}$ has a nilpotent ideal that contains any other nilpotent ideal. This is called the *nilpotent radical* of $\mathfrak{L}$, and it will be represented as $R_n(\mathfrak{L})$. Additionally, $R_n(\mathfrak{L})$ is an isolated ideal, as it is nothing but $R_n(\mathfrak{L}_K) \cap \mathfrak{L}$.

We will say that an R-Lie algebra $\mathfrak{L}$ is *splittable* if there exists an R-Lie subalgebra $\mathfrak{S} \leq \mathfrak{L}$ such that $\mathfrak{L} = R_n(\mathfrak{L}) \oplus \mathfrak{S}$, that is, $\mathfrak{L}$ is the semidirect product of $\mathfrak{S}$ with the ideal $R_n(\mathfrak{L})$. For the categoricaly minded reader we point out that this condition is equivalent to the fact that the short exact sequence

$$0 \to R_n(\mathfrak{L}) \to \mathfrak{L} \to \mathfrak{L}/R_n(\mathfrak{L}) \to 0$$

splits in the category of R-Lie algebras.

To deal with the splittable case, we can blend the preceding regular representation and the representations induced from derivations:

Definition A.11. Let A be an R-algebra. A *derivation* of A is an R-module endomorphism $D \in \mathrm{End}_R(A)$ that satisfies *Leibniz identity*, i.e.

$$D(ab) = aD(b) + D(a)b \; \forall a, b \in A.$$

The set of all derivations of A is denoted by $\mathrm{Der}_R(A)$.

For example, by virtue of Jacobi's identity, for all $x \in \mathfrak{L}$ we have that ad_x is a derivation of the R-Lie algebra $\mathfrak{L}$. Starting from a derivation D of $\mathfrak{L}$ we can induce a derivation of $\mathcal{U}_R(\mathfrak{L})$, which will be denoted by D^*. For that, we should extend D by imposing Leibniz identity, i.e. by taking the linear extension of the rule

$$D^*(x_1 \dots x_t) = \sum_{i=1}^{t} x_1 \dots x_{i-1} D(x_i) x_{i+1} \dots x_t,$$

together with $D^*(1) = 0$ as it must happen for every derivation of an algebra with identity. Actually, this extension is a consequence of the universal property (compare with [41, Chapter V, Theorem 1.1(7)]).

Lemma A.12. *Let $\mathfrak{L}$ be a nilpotent R-Lie lattice and $D \in \mathrm{Der}_R(\mathfrak{L})$. Then $D^*(\mathfrak{U}^m(\mathfrak{L})) \subseteq \mathfrak{U}^m(\mathfrak{L})$ for every $m \in \mathbb{N}$.*

Proof. Let $\{x_1, \dots, x_r\}$ be the basis of $\mathfrak{L}$ with respect to which the weight function ω is defined (compare with Subsection A.3.1). Since D is a derivation, $D(\mathfrak{L}_i) \subseteq \mathfrak{L}_i$ for all $i \in \{1, \dots, c\}$, so $\omega(D(x_i)) \geq \omega(x_i)$ for all $i \in \{1, \dots, r\}$. Hence, if $x_{i_1} \dots x_{i_t} \in \mathfrak{U}^m(\mathfrak{L})$, by (A.3)

$$\omega\left(D^*(x_{i_1} \dots x_{i_t})\right) \geq \min_{j=1,\dots,t} \left\{\omega(x_{i_1} \dots D(x_{i_j}) \dots x_{i_t})\right\} \geq \omega(x_{i_1} \dots x_{i_t}) > m. \quad \square$$

Proposition A.13 (Zassenhaus extension, cf. [11, Chapter I, § 7.3, Theorem 1] and [41, Chapter VI, Theorem 2.1])**.** *Let $\mathfrak{L}$ be a splittable R-Lie lattice and let c be the nilpotency class of $R_n(\mathfrak{L})$. Then, there exists a finite representation*

$$\Phi \colon \mathfrak{L} \to \mathrm{End}_R \left(\frac{\mathcal{U}_R(R_n(\mathfrak{L}))}{\mathfrak{U}^c(R_n(\mathfrak{L}))} \right)$$

of $\mathfrak{L}$ that is injective in $R_n(\mathfrak{L})$. In particular, $\deg \Phi$ *depends only on* $\mathrm{rk}\, R_n(\mathfrak{L})$.

Proof. Denote for simplicity $R_n(\mathfrak{L})$ as $\mathfrak{N}$. Then, $\mathfrak{L} = \mathfrak{N} \oplus \mathfrak{S}$ for some R-Lie subalgebra $\mathfrak{S} \leq \mathfrak{L}$. From Lemma A.12, $\mathrm{ad}_x^* (\mathfrak{U}^c(\mathfrak{N})) \subseteq \mathfrak{U}^c(\mathfrak{N})$ for all $x \in \mathfrak{L}$, so we can define the map

$$\Phi \colon \mathfrak{L} = \mathfrak{N} \oplus \mathfrak{S} \to \mathrm{End}_R \left({}^{\mathcal{U}_R(\mathfrak{N})}/_{\mathfrak{U}^c(\mathfrak{N})}\right), \; n + s \mapsto \ell_n + \mathrm{ad}_s^*.$$

In order to show that it is an R-Lie algebra homomorphism, it suffices to confirm that

$$\Phi\left([s, n]\right) = [\Phi(s), \Phi(n)] = [\mathrm{ad}_s^*, \ell_n]$$

for all $n \in \mathfrak{N}$ and $s \in \mathfrak{S}$. For that, note that for any $n \in \mathfrak{N}$ and any $D \in \mathrm{Der}_R\left(\mathcal{U}_R(\mathfrak{N})\right)$:

$$[D, \ell_n](u) = D \circ \ell_n(u) - \ell_n \circ D(u) = D(n)u = \ell_{D(n)}(u), \; \forall u \in \mathfrak{N}.$$

Moreover, since $\mathfrak{N}$ is an ideal, then $[s, n] \in \mathfrak{N}$, so

$$\Phi([s,n]) = \ell_{[s,n]} = \ell_{\mathrm{ad}^*_s(n)} = [\mathrm{ad}^*_s, \ell_n] = [\Phi(s), \Phi(n)] \,.$$

Consequently, Φ is an R-Lie algebra representation. In addition, its kernel has trivial intersection with the nilpotent radical, as $\Phi|_{\mathfrak{N}}$ is nothing but the faithful representation $\mathcal{L}_c$ of $\mathfrak{N}$.

Finally, from (A.4) we conclude that

$$\deg \Phi \leq \sqrt{\frac{\operatorname{rk} R_n(\mathfrak{L}) + 1}{\operatorname{rk} R_n(\mathfrak{L})}} \cdot 4^{\operatorname{rk} R_n(\mathfrak{L})}. \tag{A.5}$$

□

A.3.3 Embedding theorem

Like nilpotency in R-Lie algebras, we can define solubility: the *derived series* of an R-Lie algebra $\mathfrak{L}$ is defined recursively as

$$\mathfrak{L}^{(1)} := \mathfrak{L}, \;\; \mathfrak{L}^{(i)} := \left[\mathfrak{L}^{(i-1)}, \mathfrak{L}^{(i-1)}\right] \;\; \forall i \geq 2,$$

$\mathfrak{L}$ is said to be *soluble* when $\mathfrak{L}^{(\ell)} = \{0\}$ for some $\ell \in \mathbb{N}$, and the smallest of such integers is the *derived length* of $\mathfrak{L}$. In addition, the sum of soluble ideals is again soluble (compare with [41, Chapter I, Proposition 7.4]), and therefore, if $\mathfrak{L}$ is finitely generated, there exists a *soluble radical* of $\mathfrak{L}$, namely a soluble ideal $R_s(\mathfrak{L})$ that contains any other soluble ideal. Obviously, any nilpotent R-Lie algebra is soluble and thus $R_n(\mathfrak{L}) \leq R_s(\mathfrak{L})$.

According to *Levi's Theorem* (see [41, Chapter III, Section 9]), if R is a field of characteristic zero, every R-Lie algebra $\mathfrak{L}$ splits as $R_s(\mathfrak{L}) \oplus \mathfrak{S}$ for some semisimple Lie subalgebra $\mathfrak{S} \leq \mathfrak{L}$, called *Levi factor* of $\mathfrak{L}$. This decomposition plays a fundamental rôle in the majority of proofs of Ado's Theorem for fields, as they firstly obtain a finite faithful representation of $R_s(\mathfrak{L})$, and then it is extended to $\mathfrak{L}$ using Zassenhaus extension (compare with Propositon A.13).

However, Levi's Theorem does not hold for general R-Lie algebras: the $\mathbb{Z}$-Lie algebra $\mathfrak{sl}_2(2\mathbb{Z}) \oplus \mathfrak{t}_2(2\mathbb{Z})$, i.e. the direct sum of 2×2 matrices of trace 0 and 2×2 upper triangular matrices over the ring $2\mathbb{Z}$, is not decomposable in the desired way (see [19, Example in pg. 838]).

In this third and last step we shall prove the main theorem. For that, we will embed the initial R-Lie lattice $\mathfrak{L}$ in a splittable R-Lie lattice and make use of the previous subsection. Actually the problem reduces to fields:

Proposition A.14 (cf. [19, Step 2 in the proof of Theorem 2])**.** *Let $\mathfrak{L}$ be an R-Lie lattice. Suppose that $\mathfrak{L}_K$ embeds in an splittable K-Lie algebra $\mathfrak{L}'_K$. Then $\mathfrak{L}$ embeds in an splittable R-Lie lattice $\bar{\mathfrak{L}}$ such that*

(i) $R_n(\mathfrak{L}) \leq R_n(\bar{\mathfrak{L}})$ *and*

(ii) $\operatorname{rk} R_n(\mathfrak{L}) = \operatorname{rk} R_n(\bar{\mathfrak{L}})$.

Proof. Denote for simplicity $\mathfrak{N}'_K = R_n(\mathfrak{L}'_K)$, and suppose that $\mathfrak{L}'_K = \mathfrak{N}'_K \oplus \mathfrak{S}'_K$ is splittable. Observe in passing that $R_n(\mathfrak{L}) \subseteq R_n(\mathfrak{L}_K) \subseteq R_n(\mathfrak{L}'_K)$.

Let $\mathfrak{S}$ be the projection of $\mathfrak{L}$ into $\mathfrak{S}'_K$, that is,

$$\mathfrak{S} = \{s \in \mathfrak{S}'_K \mid \exists\, x \in \mathfrak{L} \text{ such that } x = n + s \text{ for some } n \in \mathfrak{N}'_K\}.$$

It is a routine exercise to verify that $\mathfrak{S}$ is an R-Lie algebra. Indeed, let $x_1 = n_1 + s_1$ and $x_2 = n_2 + s_2 \in \mathfrak{L}$, where $n_i \in \mathfrak{N}'_K$ and $s_i \in \mathfrak{S}'_K$ ($i \in \{1, 2\}$), then

$$[x_1, x_2] = [n_1, x_2] + [s_1, n_2] + [s_1, s_2], \tag{A.6}$$

where $[x_1, x_2] \in \mathfrak{L}$, $[n_1, x_2] + [s_1, n_2] \in \mathfrak{N}'_K$ and $[s_1, s_2] \in \mathfrak{S}'_K$. Therefore, since $\mathfrak{S}$ is finitely generated, there exists $\lambda \in R \setminus \{0\}$ such that $\lambda\mathfrak{S} \subseteq \mathfrak{L}$.

Let $s = \operatorname{rk} R_n(\mathfrak{L})$ and let c be the nilpotency class of $\mathfrak{N}'_K$. Define the isolated ideals $\mathfrak{N}_i := \gamma_i(\mathfrak{N}'_K) \cap \mathfrak{L} \trianglelefteq \mathfrak{L}$, $i \in \{1, \ldots, c\}$. Moreover, $\mathfrak{N}_i \trianglelefteq \mathfrak{N}_1 = R_n(\mathfrak{L})$, and, consequently, we can choose an R-basis $\{x_1, \ldots, x_s\}$ for $R_n(\mathfrak{L})$ such that the first s_1 elements constitute a basis for $\mathfrak{N}_c$, the first s_2 elements ($s_2 \geq s_1$) constitute a basis for $\mathfrak{N}_{c-1}$ and so forth.

Now, define the R-module

$$\mathfrak{N} := \bigoplus_{i=1}^{c} \frac{1}{\lambda^i} \mathfrak{N}_i \subseteq R_n(\mathfrak{L}_K),$$

which is a free R-module of rank s, as it has basis

$$\left\{ \frac{1}{\lambda^i} x_j \mid 1 \leq i \leq c,\ s_{c-i} \leq j \leq s_{c-i+1} \right\}$$

(with the conventions $s_0 = 1$ and $s_c = s$). Since

$$\left[\frac{1}{\lambda^i} \mathfrak{N}_i, \frac{1}{\lambda^j} \mathfrak{N}_j \right] = \frac{1}{\lambda^{i+j}} [\mathfrak{N}_i, \mathfrak{N}_j] \leq \frac{1}{\lambda^{i+j}} \mathfrak{N}_{i+j},$$

we conclude that $\mathfrak{N}$ is a nilpotent R-Lie lattice.

Now, consider $\bar{\mathfrak{L}} := \mathfrak{N} \rtimes \mathfrak{S}$. We claim that $\bar{\mathfrak{L}}$ is an R-Lie lattice that contains $\mathfrak{L}$. Indeed, note that $\mathfrak{S} = \mathfrak{L} + {}^1\!/\!_\lambda \mathfrak{N}_1$, and thus $\mathfrak{L} \subseteq \bar{\mathfrak{L}}$. Moreover,

$$\begin{aligned}\left[\frac{1}{\lambda^i}\mathfrak{N}_i, \mathfrak{S}\right] &\leq \left[\frac{1}{\lambda^i}\mathfrak{N}_i, \mathfrak{L} + \frac{1}{\lambda}\mathfrak{N}_1\right] \leq \frac{1}{\lambda^i}\left[\mathfrak{N}_i, \mathfrak{L}\right] + \frac{1}{\lambda^{i+1}}\left[\mathfrak{N}_i, \mathfrak{N}_1\right] \\ &\leq \frac{1}{\lambda^i}\mathfrak{N}_i + \frac{1}{\lambda^{i+1}}\mathfrak{N}_{i+1} \leq \mathfrak{N}\end{aligned}$$

for every $i \in \{1, \ldots, c\}$, so $\bar{\mathfrak{L}}$ is an R-Lie lattice. Finally, we observe that the nilpotent radical of $\bar{\mathfrak{L}}$ is $\mathfrak{N}$, which contains $R_n(\mathfrak{L})$. Thus, $\operatorname{rk} R_n(\bar{\mathfrak{L}}) = \operatorname{rk} R_n(\mathfrak{L}) = s$ and, by construction, $\bar{\mathfrak{L}}$ is splittable. □

Furthermore, in view of Levi's Theorem whenever R is a field of characteristic zero the R-Lie algebra $\mathfrak{L}$ is splittable provided $R_n(\mathfrak{L}) = R_s(\mathfrak{L})$. Over general PIDs, the situation is unclear. Nonetheless, $\mathfrak{L}$ can be embedded in an splittable R-Lie algebra, more concretely:

Corollary A.15. *Let $\mathfrak{L}$ be an R-Lie lattice and assume that $R_n(\mathfrak{L}) = R_s(\mathfrak{L})$. Then $\mathfrak{L}$ embeds in an splittable R-Lie lattice $\bar{\mathfrak{L}}$ such that $R_n(\mathfrak{L}) \leq R_n(\bar{\mathfrak{L}})$ and* $\operatorname{rk} R_n(\mathfrak{L}) = \operatorname{rk} R_n(\bar{\mathfrak{L}})$.

Proof. Since $R_n(\mathfrak{L}_K) = R_s(\mathfrak{L}_K)$, then $\mathfrak{L}_K$ is splittable in view of Levi's Theorem. □

Finally, we have the desired general embedding, due by Neretin [60] (the following proof uses some involved results of the theory of Lie algebras, so we shall briefly sketch it).

Theorem A.16 (Embedding Theorem)**.** *Let $\mathfrak{L}$ be an R-Lie lattice. There exists an splittable R-Lie lattice $\bar{\mathfrak{L}}$ extending $\mathfrak{L}$ such that $R_n(\mathfrak{L}) \leq R_n(\bar{\mathfrak{L}})$ and* $\operatorname{rk} R_n(\bar{\mathfrak{L}}) = \operatorname{rk} R_s(\mathfrak{L})$.

Proof. In view of Proposition A.14, it suffices to prove the existence of an splittable K-Lie algebra $\mathfrak{L}'_K$ extending $\mathfrak{L}_K$ such that $\dim_K R_n(\mathfrak{L}'_K) = \operatorname{rk} R_s(\mathfrak{L})$.

Suppose that $R_n(\mathfrak{L}) = \langle x_1, \ldots, x_s \rangle_R$ and let $\mathfrak{S}_K = \langle z_1, \ldots, z_l \rangle_K$ be a Levi factor of $\mathfrak{L}_K$. Then, there exist some elements $y_1, \ldots, y_k \in R_s(\mathfrak{L}_K)$, where $k = \operatorname{rk} R_s(\mathfrak{L}) - s$, such that

$$\mathfrak{L}_K = Kx_1 \oplus \cdots \oplus Kx_s \oplus Kz_1 \oplus \cdots \oplus Kz_l \oplus Ky_1 \oplus \cdots \oplus Ky_k.$$

Furthermore, according to [41, Chapter II, Theorem 7.13],

$$[R_s(\mathfrak{L}_K), \mathfrak{L}_K] \leq R_n(\mathfrak{L}_K). \tag{A.7}$$

Therefore,

$$\mathfrak{I}_i := \langle x_1, \dots, x_s, z_1, \dots, z_h \rangle \oplus K y_1 \oplus \cdots \oplus K y_i$$

is an ideal for every $i \in \{0, \dots, k\}$. In particular, $\mathfrak{L}_K = \mathfrak{I}_k$ is an $\mathfrak{S}_K$-algebra. Since $\mathfrak{S}_K$ is semisimple and $\operatorname{char} K = 0$, according to Weyl's Theorem on complete reducibility (see [41, Chapter III, Theorem 7.8]), $\mathfrak{I}_k$ is a semisimple $\mathfrak{S}_K$-module and so there exists a principal $\mathfrak{S}_K$-module $K\tilde{y}_1$ such that

$$\mathfrak{L}_K = \mathfrak{I}_{k-1} \oplus K\tilde{y}_1.$$

Now, $K\tilde{y}_1$ is an $\mathfrak{S}_K$-module, but at the same time by (A.7), $[K\tilde{y}_1, \mathfrak{S}_K] \leq R_n(\mathfrak{L}_K)$, and thus $[K\tilde{y}_1, \mathfrak{S}_K] = 0$.

Furthermore, since $\mathrm{ad}_{\tilde{y}_1} \in \mathrm{Der}_K(\mathfrak{L}_K)$ and K has characteristic zero, according to the Jordan-Chevalley decomposition (see [41, Chapter III, Theorem 11.17] and [63, Proposition 3]), there exist a nilpotent derivation $d_{n,1}$ (i.e. $d_{n,1}^C = 0$ for some $C \in \mathbb{N}$) and a semisimple derivation $d_{s,1}$ (i.e. $d_{s,1}$ is a diagonalizable operator over the algebraic closure K^{alg}) such that

$$\mathrm{ad}_{\tilde{y}_1} = d_{n,1} + d_{s,1}.$$

Additionally, these derivations are unique and they commute.

Consider the K-Lie algebra

$$\mathfrak{L}_1 = \langle x_1, \dots, x_s, x_1' \rangle_K \oplus \langle z_1, \dots, z_l, z_1' \rangle_K,$$

where x_1' and z_1' are simply formal symbols, endowed with the following Lie bracket:

$$[u, v]_{\mathfrak{L}_1} = [u, v]_{\mathfrak{L}_K} \qquad [u, x_1']_{\mathfrak{L}_1} = d_{n,1}(u)$$

$$[u, z_1']_{\mathfrak{L}_1} = d_{s,1}(u) \qquad \text{and } [x_1', z_1']_{\mathfrak{L}_1} = 0,$$

for all $u, v \in R_n(\mathfrak{L}_k) \rtimes \mathfrak{S}_K$.

Observe that the assignation $\tilde{y}_1 = x'_1 + z'_1$ defines a K-Lie algebra monomorphism $\mathfrak{L}_K \hookrightarrow \mathfrak{L}_1$. Moreover, since $\mathrm{ad}_{x'_1}$ is a nilpotent derivation then

$$\mathfrak{N}_1 = \langle x_1, \ldots, x_s, x'_1 \rangle = R_n(\mathfrak{L}_K) \oplus K x'_1$$

is a nilpotent Lie algebra. In addition, since $\tilde{y}_1$ commutes with $\mathfrak{S}_K$, so does z'_1 and therefore the Lie algebra

$$\mathfrak{S}_1 = \mathfrak{S}_K \oplus K z'_1$$

is reductive, as it is the sum of a nilpotent and an abelian Lie algebra. Besides, since $\mathrm{ad}_{z'_1}$ is a semisimple operator and the action of $\mathfrak{S}_K$ in $\mathfrak{L}_1$ is completely reducible, the action of $\mathfrak{S}_1$ in $\mathfrak{L}_1$ is also completely reducible. In particular, since $\mathfrak{I}_{k-2} \trianglelefteq \mathfrak{I}_{k-1} \trianglelefteq \mathfrak{L}_1$, there exists a principal $\mathfrak{S}_1$-module $K\tilde{y}_{k-1}$ such that

$$\mathfrak{I}_{k-1} = \mathfrak{I}_{k-2} \oplus K\tilde{y}_{k-1}.$$

Repeating the preceding argument for the derivation $\mathrm{ad}_{\tilde{y}_{k-1}} \in \mathrm{Der}_K(\mathfrak{L}_1)$ there exists a K-Lie algebra

$$\mathfrak{L}_2 := \langle x_1, \ldots, x_s, x'_1, x'_2 \rangle_K \oplus \langle z_1, \ldots, z_l, z'_1, z'_2 \rangle_K$$

such that $\mathfrak{N}_2 = \mathfrak{N}_1 \oplus K x'_2$ is nilpotent, and $\mathfrak{S}_2 = \mathfrak{S}_1 \oplus K z'_2$ is reductive and acts completely reducibly on $\mathfrak{L}_2$.

Repeating the preceding procedure k times, eventually we obtain an splittable K-Lie algebra $\mathfrak{L}_k = \mathfrak{N}_k \oplus \mathfrak{S}_k$, where $\mathfrak{N}_k = \langle x_1, \ldots, x_s, x'_1, \ldots, x'_k \rangle_K$ is the nilpotent radical and $\mathfrak{S}_k = \langle z_1, \ldots, z_l, z'_1, \ldots, z'_k \rangle_K$ is a reductive K-Lie algebra. Furthermore, $\mathfrak{L}_k$ extends $\mathfrak{L}_K$ with the correspondence $\tilde{y}_j = x'_j + z'_j$. Note that $\dim_K \mathfrak{N}_k = s + k = \mathrm{rk}\, R_s(\mathfrak{L})$. □

Therefore, we gather all the ingredients to obtain the principal result:

proof of Theorem A.4. Let $\mathfrak{L}$ be a R-Lie lattice of rank r. There exists an splittable R-Lie lattice $\bar{\mathfrak{L}}$ extending $\mathfrak{L}$ such that $R_n(\mathfrak{L}) \le R_n(\bar{\mathfrak{L}})$ and $\mathrm{rk}\, R_n(\bar{\mathfrak{L}}) = R_s(\mathfrak{L})$. Proposition A.13 provides an R-Lie algebra representation Φ of $\bar{\mathfrak{L}}$ which is injective in $R_n(\bar{\mathfrak{L}})$ and whose degree is bounded by $f(\mathrm{rk}\, R_n(\bar{\mathfrak{L}}))$ for the non-decreasing function $f\colon \mathbb{N}_0 \to \mathbb{N}$, $r \mapsto \sqrt{\frac{r+1}{r}} \cdot 4^r$.

Therefore $\tilde{\Phi} := \Phi|_{\mathfrak{L}} \oplus \mathrm{Ad}$ is an R-Lie algebra representation of $\mathfrak{L}$ that is faithful, as

$$\ker \tilde{\Phi} = \ker \Phi|_{\mathfrak{L}} \cap \ker \mathrm{Ad} \subseteq (\mathfrak{L} \setminus R_n(\mathfrak{L})) \cap Z(\mathfrak{L}) = \{0\}.$$

Thus, since $\mathrm{rk}\, R_n\left(\bar{\mathfrak{L}}\right) = \mathrm{rk}\, R_s(\mathfrak{L})$, then

$$\deg \mathfrak{L} \le \deg \tilde{\Phi} \le f(\mathrm{rk}\, R_s(\mathfrak{L})) + r \le f(r) + r.$$

Finally, the (A.5) gives the precise bound. □

A.4 Notes

The original results of this appendix are collected in [77].

We shall finish the appendix with a couple of remarks. Firstly, Harish-Chandra gave in [34] yet another proof of Ado's Theorem, and improved the result by observing that we could construct a finite faithful representation that maps the elements of the nilpotent radical to nilpotent endomorphisms –$f \in \mathrm{End}_R(W)$ is nilpotent if there exists $C \in \mathbb{N}$ such that $f^C = 0$–. Such representations are called *nil-representation*s, and it is worth pointing out that all the representations that appear throughout the appendix are nil-representation. Indeed, the adjoint representation is a nil-representation in view of Jacobi's identity, and for every $x \in R_n(\mathfrak{L})$ we know that

$$x^{c+1} \in \mathfrak{U}^c(R_n(\mathfrak{L})),$$

so

$$\Phi(x)^{c+1} \equiv \ell_x^{c+1} = \ell_{x^{c+1}} \equiv 0 \mod \mathfrak{U}^c(R_n(\mathfrak{L})).$$

That is, $\mathfrak{L}_c$ is an nil-representation.

Secondly, by using more involved combinatorial arguments, it is possible to prove that

$$\mathrm{rk}\left(\frac{\mathcal{U}_R(\mathfrak{L})}{\mathfrak{U}^c(\mathfrak{L})}\right) < \alpha\frac{2^r}{\sqrt{r}},$$

where

$$\alpha = \sqrt{\frac{2}{\pi}}\prod_{j=1}^{\infty}\frac{2^j}{2^j - 1} \in (2.762, 2.763)$$

(compare with [13, Lemma 5 (3)]). However, for our goals, chiefly the application in Chapter 2, the bound of Theorem A.4 is good enough. Nonetheless, it is worth a remark that over general PIDs, we could retrieve (A.1), the best bound yet known for fields.

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Part II

Talde R-analitiko profinituak

Konbentzioak. Eraztun guztiak trukakorrak eta identitatedunak dira. Gainera, tesian zehar p beti zenbaki lehen bat izanen da, eta q zenbaki lehen baten berretura.

Notazioa. Notazio gehiena estandarrra da, $A^{(n)}$ izan ezik. Horrek A multzoaren $n$garren berretura kartesiarra adierazten du (notazio hori erabiliko da $\mathfrak{a}$ idealaren $\mathfrak{a}^{(n)}$ berretura kartesiarra eta $\mathfrak{a}^n$ berretura ideala bereizteko, izan ere, maiz $(\mathfrak{a}^n)^{(m)}$ modukoak agertuko dira). Era berean, $f\colon A \to B$ funtzioa emanda, $f^{(n)}$ moduan izendatuko dugu $A^{(n)} \to B^{(n)}$, $(a_1, \ldots, a_n) \mapsto (f(a_1), \ldots, f(a_n))$ funtzioa.

Gainerako terminologia honakoa da:

$\mathbb{N}$	zenbaki arruntak
$\mathbb{N}_0$	zenbaki arruntak 0-rekin
$\mathbb{Z}$	zenbaki osoak
$\mathbb{Z}_p$	zenbaki oso p-adikoak
$\mathbb{Q}$	zenbaki arrazionalak
$\mathbb{Q}_p$	zenbaki p-adikoak
$\mathbb{R}$	zenbaki errealak
$\mathbb{R}_{\geq 0}$	zenbaki erreal ez-negatiboak
$\mathbb{C}$	zenbaki konplexuak
$\mathbb{F}_q$	q tamainako gorputz finitua
$\log_a$	a oinarriko logaritmoa
$\mathcal{P}(A)$	A-ren parteak
$A \times B$	A eta B-ren biderketa kartesiarra
$\prod_{i \in I} A_i$	$\{A_i\}_{i \in I}$ multzo zuzenduaren biderketa kartesiarra
$A^{(k)}$	A-ren $k$garren berretura kartesiarra
$A \oplus B$	A eta B-ren batura zuzena
$\bigoplus_{i \in I} A_i$	$\{A_i\}_{i \in I}$ multzo zuzenduaren batura zuzena
$A \ltimes B$	A eta B-ren biderketa erdizuzena
$\prod_{i \in I} A_i / \mathcal{U}$	$\{A_i\}_{i \in I}$ familiaren ultrabiderketa $\mathcal{U}$ ultrafiltroarekin
$A^{\mathcal{U}}$	A-ren ultraberretura $\mathcal{U}$ ultrafiltroarekin

$A \subseteq B$	A B-ren azpimultzoa da
$A \subseteq_o B$	A B-ren azpimultzo irekia da
$A \subseteq_c B$	A B-ren azpimultzo itxia da
$A \leq B$	A B-ren azpitaldea da
$A \leq_o B$	A B-ren azpitalde irekia da
$A \leq_c B$	A B-ren azpitalde itxia da
$A \trianglelefteq B$	A B-ren azpitalde normala da
$A \trianglelefteq_o B$	A B-ren azpitalde normal irekia da
$A \trianglelefteq_c B$	A B-ren azpitalde normal itxia da
$A \operatorname{char} B$	A B-ren azpitalde karakteristikoa da

Izan bitez G taldea eta $g, x, y \in G$:

x^y	$y^{-1}xy$
$[x, y]$	$x^{-1}y^{-1}xy$
$[x_1, \ldots, x_n]$	$[[x_1, \ldots, x_{n-1}], x_n]$
$Z(G)$	G-ren zentroa
$C_G(g)$	$g \in G$-ren zentralizatzailea
$G' = [G, G]$	G-ren azpitalde deribatua: $\langle [x, y] \mid x, y \in G \rangle$.
G^n	$n$garren berretura azpitaldea: $\langle g^n \mid g \in G \rangle$
$[H_1, \ldots, H_n]$	$\langle [h_1, \ldots, h_n] \mid h_i \in H_i \rangle$
c_y	konjokazio isomorfismoa: $G \to G$, $x \mapsto x^y$
L_y	ezker biderketa funtzioa $G \to G$, $x \mapsto yx$
R_y	eskuin biderketa funtzioa $G \to G$, $x \mapsto xy$

$\ker f$	f talde (eraztun) homomorfismoaren nukleoa
$\operatorname{im} f$	f talde (eraztun) homomorfismoaren irudia
$\operatorname{Hom}(A, B)$	$f \colon A \to B$ talde (eraztun) homomorfismo guztiak

Izan bedi Q eraztuna:
Izan bedi K gorputza:

K^{alg}	K-ren itxitura aljebraikoa
$\dim_K$	K-bektore espazio dimentsioa

Izan bedi M barietate R-analitikoa:

$\mathcal{U}(Q)$	Q-ren unitateak
$\operatorname{char} Q$	Q-ren karakteristika
$\dim_{\mathrm{Krull}} Q$	Q-ren Krullen dimentsioa
$\mathrm{Frac}(Q)$	Q-ren zatikien gorputza (Q integritate domeinua denean)
$Q[[t_1, \ldots, t_m]]$	koefizienteak Q-n dituzten berretura serie formalen eraztuna

$\mathrm{M}_{n\times m}(Q)$	koefizienteak Q-n dituzten $n \times m$ tamainako matrizeak
$\mathrm{M}_n(Q)$	koefizienteak Q-n dituzten $n \times n$ tamainako matrizeak
$\mathrm{GL}_n(Q)$	koefizienteak Q-n dituen talde lineal orokorra
$\mathrm{SL}_n(Q)$	koefizienteak Q-n dituen talde lineal berezia
$\mathrm{SO}_n(Q)$	koefizienteak Q-n dituen talde ortogonal berezia
$\mathrm{Sp}_n(Q)$	koefizienteak Q-n dituen talde sinplektikoa

$\dim_x M$	M barietatearen dimentsio analitikoa x puntuan
$\dim M$	M barietate puruaren dimentsio analitikoa

Izan bedi $\mathcal{G}$ talde aljebraiko lineala:

$R_u(\mathcal{G})$	erradikal unipotentea

rk	matrize baten heina
res. rk	matrize baten hondar-heina
det	matrize baten determinantea
tr	matrize baten aztarna

$D\mathbf{U}$	$\mathbf{U}$ berretura serie tuplaren diferentziala
$\mathcal{J}_x F$	F-ren matrize jacobirarra x puntuan

$\operatorname{rk} M$	M modulu askearen heina
$\mathrm{Iso}_M(N)$	N-ren isolatzailea M-n
$\mathrm{End}_R(M)$	M moduluaren R-endomorfismoak

hdim	Hausdorffen dimentsioa
hspec	Hausdorffen espektroa
hdim_{st}	Hausdorffem dimentsio estandarra
hspec_{st}	Hausdorffen espektro estandarra
bdim	kutxa-dimentsioa
bdim_{st}	kutxa-dimentsio estandarra
lbdim	behe kutxa-dimentsioa
lbdim_{st}	behe kutxa-dimentsio estandarra
ubdim	goi kutxa-dimentsioa
ubdim_{st}	goi kutxa-dimentsio estandarra

$w\{G\}$	G-ren w-balio multzoa
$w(G)$	w-ren hitzezko azpitaldea G-n
$w^*(G)$	w-ren azpitalde marjinala G-n
$X^{*\ell}$	$X \cup X^{-1} \cup \{1\}$ multzoko ℓ elementuren biderketak osatutako multzoa

$\deg \mathfrak{L}$	$\mathfrak{L}$ Lie aljebraren maila
$\deg \phi$	ϕ Lie adierazpenaren maila
$Z(\mathfrak{L})$	$\mathfrak{L}$ Lie aljebraren zentroa
$R_n(\mathfrak{L})$	$\mathfrak{L}$ aljebraren erradikal nilpotentea
$R_s(\mathfrak{L})$	$\mathfrak{L}$ aljebraren erradikal ebazgarria
$\mathbf{T}_R(\mathfrak{L})$	$\mathfrak{L}$ aljebraren tortsio aljebra
$\mathcal{U}_R(\mathfrak{L})$	$\mathfrak{L}$ aljebraren inguratze aljebra unibertsala
$\text{Der}_R(\mathfrak{L})$	$\mathfrak{L}$ aljebraren deribazioak
$\text{Cent}(\mathfrak{L})$	$\mathfrak{L}$ aljebraren zentroidea

"I tend to think too much.
My greatest successes came from decisions I made when I stopped thinking and simply did what felt right.
Even if there was no good explanation for what I did. [...]
Even if there were very good reasons for me not to do what I did."

Kvothe, (Patrick Rothfuss, *The Name of the Wind*)

Sarrera

Tesi hau talde analitikoen inguruko monografikoa da. Talde horiek eraztun topologiko egoki baten gainean barietate analitikoak ere badiren talde abstraktuak dira, bi egitura horiek bateragarriak izanik. Hots, biderketa funtzioa eta alderantzizko funtzioa analitikoak dira.

Gorputz topologikoen gaineko talde analitikoen teoria talde profinituen adibide-iturburua da. Noski, $\mathbb{R}$ eta $\mathbb{C}$ gorputz klasikoen gaineko barietateak, finituak izan ezean, ez dira talde profinituak; aldi berean trinkoak eta $\mathbb{C}^{(n)}$-ko azpimultzo guztiz diskonexu bati homeomorfoak izan behar baitute. Aitzitik, oinarriko eraztuna bere kabuz, talde abeldar gisa, talde profinitua denean, talde analitiko profinituak ager daitezke. Esate baterako, Lazardek, *Groupes analytiques p-adiques* [50] tratatuan, talde p-adiko analitikoak, hau da, $\mathbb{Q}_p$ zenbaki p-adikoen –baliokideki, $\mathbb{Z}_p$ zenbaki oso p-adikoen– gaineko Lieren talde analitikoak aurkeztu zituen; eta talde p-adiko analitiko trinkoak talde profinituak direla frogatu zuen.

Jatorrizko ikuspegi analitiko hutsetik at, talde p-adiko analitikoen karakterizazio anitz daude ([24, Apendizea A]-n topa daiteke zerrenda zehatza). Horien artean, Lazard berak *talde p-adiko analitikoetarako Hilberten 5. problema* dei dezakeguna ebazteko erakutsi zuen talde p-adiko trinkoak hain justu birtualki heina finituko pro-p taldeak direla. Alegia, indize finituko pro-p azpitalde bat duten talde profinituak dira non pro-p azpitalde horren azpitalde guztiak finituki sortuak diren eta haien sortzaile kopurua bornatuta dagoen. Geroztik, talde hauen inguruan teoria emankorra garatu da, eta hainbat propietate interesgarri aurkitu dira: guztiek, $\mathbb{Z}_p$ zenbaki p-adiko osoen taldeak izan ezik, *Golod-Shafarevichen ezberdintza* betetzen dute (Lubotzky [52]), *azpitalde hazkunde polinomiala* dute

(Lubotzky eta Mann [53]), hitz guztiek *zabalera* finitua dute (Jaikin-Zapirain [44]), etab.

Orobat, R eraztun topologiko orokor batetik abiatu, eta haren gainean barietate egitura modu berean definituz talde p-adikoen nozioa orokortu daiteke. Hots, R-ren gaineko talde analitikoak, aurrerantzean *talde R-analitikoak* deituko ditugunak, defini daitezke. Hala, Bourbakik (edo hobeki esanda bourbakistek) [11] eta Serrek [68] $\mathbb{F}_p((t))$ gorputz lokalaren gaineko talde analitikoak, hau da, talde p-adikoen parekoak karakteristika positiboan, aztertu zituzten. Areago, bigarren edizioa argitaratzean autoreek *Analytic pro-p groups* [24] liburuari hamahirugarren kapitulu bat gehitu zioten. Bertan *pro-p domeinu* orokorren gainean talde analitikoak definitu zituzten, eta, horrenbestez, teoria orokorrago horren lehenbiziko urratsak eman zituzten. Gogora dezagun pro-p domeinu bat *R integritate-domeinu lokal noetherdar bat dela, osoa ideal maximalak definitzen duen metrikarekiko eta p karakteristikako hondar-gorputz finitua duena* (Azpiatala 1.1.1-en eraztun horiek eta haien definizioan ageri diren kontzeptuak azaltzen dira). Talde orokorrago horiek propietate aljebraiko onak dituzte (ikusi [14], [42], [43], [45] edota [54]); ez, ordea, talde p-adiko analitikoenak bezain onak. Bereziki, ez dago, ezta aieru mailan ere, talde analitiko horien karakterizaziorik talde teoriako termino hutsetan.

Tesi honen xedea teoria horri jarraipena ematea da. Alde batetik, talde R-analitikoen azterketa sistematikoarekin jarraitzen dugu, eta, horrenbestez, tesi hau nolabait [24, Kapitulua 13]-ren jarraipena da. Bestetik, teoria hori ikerketa emaitza berriekin hornitzen da. Horietako gehienek talde p-adiko analitikoetarako jada ezagunak ziren propietateak talde R-analitiko orkorragoetara hedatzen dituzte; areago, usuenik kasu p-adikoa frogen funtsezko osagaia da.

Aipa dezagun edukia kapituz kapitulu xehetasun gehiagorekin: **Kapitulua 1**en, talde R-analitikoak aurkezten dira, arestian aipatutako [24, Kapitulua 13]-n oinarrituz. Arreta berezia jarriko zaie talde R-estandarrei, ziurrenik talde R-analitikoen adibide nagusia, eta horiei elkar dakiokeen Lieren aljebrari. Halaber, kapituluan zehar gerora erabiliko dugun makineria garatuko dugu, talde R-analitikoen inguruko oinarrizko kontzeptu asko oraindik literaturan garatzeke baitzeuden. Besteak beste, [24, 349. orria]-n autoreek azpimarratzen dute "*kapitulu honetako talde analitiko orokorragoetarako, kontzeptu horiek [azpibarietateak eta zatidura barietateak] oraindik garatzeke daude*"*, eta horixe da 1.5 atalean

*Jatorrizko esaldia ingelesezko bertsioan topa daiteke.

egiten dena.

Kapitulua 2 linealtasunari buruzkoa da. Zehazkiago, *R zero karakteristikako pro-p domeinua denean, talde R-analitiko trinkoak linealak direla* frogatuko dugu, hau da, $\mathrm{GL}_n(K)$ *talde linear orokorrean* murgildu daitezkeela K gorputz batentzat. Horrek Lubotzky eta Shaleven galdera bati (ikusi [54, 311 orriko 2. galdera]) erantzun partziala ematen dio. Emanen dugun froga talde p-adiko analitikoen linealtasunean oinarritzen da, eta eredu teoriako kutsua du. Izan ere, talde R-analitikoa $\mathrm{GL}_n(\mathbb{Z}_p)$-ren ultraberretura egoki batean murgiltzen dugu. Hala ere, eredu teoria ez da tesiko gai nagusietako bat, eta, hortaz, emanen dugun froga, ahal den heinean, eredu teorian aurretiazko ezagutzarik izan gabe ulertzeko modukoa da.

Kapitulua 3 Hausdorffen dimentsioaren ingurukoa da. Ikusiko dugu talde R-analitiko trinko batean taldearen egitura analitikoa aintzat hartzen duen metrika bat dagoela, eta metrika horrekiko Hausdorffen dimentsioa, $\mathrm{hdim}\colon \mathcal{P}(G) \to [0,1]$ dimentsio fraktal bat, nola definitu gogoratuko dugu. Kapituluak bi zati nagusi ditu. Lehenik eta behin, azpibarietate itxi baten dimentsio analitikoaren eta Hausdorffen dimentsioaren arteko lotura aztertuko dugu. Azterlan hau Fernández-Alcober, Gianelli eta González-Sánchezen [27] artikuluan oinarritzen da. Bigarrenik, batik bat talde $\mathbb{F}_p[[t]]$-analitikoetan jarriko dugu arreta, eta horien Hausdorffen espektroa, hau da,

$$\mathrm{hspec}(G) := \{\mathrm{hdim}(H) \mid H \leq G \text{ azpitalde itxia da}\}$$

multzoa, deskribatuko dugu.

Kapitulua 4-n hitzak aztertuko ditugu. Hitz bat k aldagaitan k sortzaileko $F(x_1,\ldots,x_k)$ talde askeko $w = w(x_1,\ldots,x_k)$ elementu bat baino ez da, eta G taldea emanda $w\colon G^{(k)} \to G$ aplikazioa definitzen du modu naturalean, $(g_1,\ldots,g_k)$-ren irudia w-n x_i-ren ordez g_i ordezkatuta lortzen den G-ko elementua delarik. Talde R-analitikoen testuinguruan, P. Hallek [33] proposatutako hitzen inguruko problemak landuko ditugu. Bertako terminolgia erabiliz, *talde R-analitiko trinkoetan hitz guztiak laburrak direla* frogatuko dugu, hau da, *G talde R-analitiko trinkoan w funtzioaren irudia, $\mathrm{im}\, w \subseteq G$, finitua bada, $w(G) = \langle \mathrm{im}\, w\rangle$ azpitalde berbala ere finitua da.*

Azkenik **Apendizea A**-n Adoren Teoremaren aldaera bat frogatuko da, ideal nagusietako domeinu baten gainean modulu askeak diren Lieren aljebretarako. Izan ere, Kapitulua 2-n teoremaren bertsio hori behar da.

Kapitulu bakoitzaren amaieran, *Oharrak* atala dago. Horietan, egilearen ekarpen originalak zehazten dira, eta beste askotariko iruzkinak egiten dira. Eskuizkribu hau autonomoa izatea nahi denez, kontzeptu gehienak xehetasunez aurkezten dira. Hala ere, talde profinituen eta pro-p talde inguruan ezagutza jakintzat ematen da, eta, behar izanez gero, irakurleak [24, Kapitulua 1]-era jo dezake.

Las frases de efecto [...] son una plaga maligna. Empezar por el principio, como si ese principio fuese la punta siempre visible de un hilo mal enrollado del que basta tirar y seguir tirando para llegar a la otra punta [...], como si [...] hubiésemos tenido en las manos un hilo liso y continuo del que no ha sido preciso dehacer nudos ni desenredar marañas.

(José Saramago, *La caverna*)

1

Talde R-analitikoak

Tesi hau talde R-analitikoei buruz da. Lehenbiziko kapitulu honetan talde horiek sakonki aurkeztuko dira, eta haien inguruko teoria garatuko da, datozen kapiluetan erabiltzeko asmoz.

1.1 Pro-p domeinuak

Lehenik eta behin, azter ditzagun barietateen koefiziente eraztunak.

Eraztun lokal bat $\mathfrak{m}$ ideal maximal bakarra duen R eraztuna da. Laburtasunagatik eraztun lokalak usuenik $(R, \mathfrak{m})$ moduan adieraziko ditugu, eta haien *hondar-gorputza* $R/\mathfrak{m}$ zatidura gorputza da. Areago, $(R, \mathfrak{m})$ integritate domeinu noetherdar lokala bada, Krullen Ebakidura Teoremaren arabera (ikusi, esate baterako, [3, Korolarioa 10.18]), $\left\{\mathfrak{m}^N\right\}_{N\in\mathbb{N}}$ familia, ideal maximalaren berretura idealak, R-ren filtrazio serie bat da, hau da, $\bigcap_{N\in\mathbb{N}} \mathfrak{m}^N = \{0\}$ da. Hori dela eta, R topologia batez horni daiteke, *topologia* $\mathfrak{m}$-*adikoa* deituko duguna. Topologia hori $||\cdot||\colon R \to \mathbb{R}_{\geq 0}$ norma baten bidez definitzen da non $||0|| = 0$ eta

$$||x|| = c^{-N} \text{ non } x \in \mathfrak{m}^N \setminus \mathfrak{m}^{N+1}$$

diren, $c > 1$ edozein zenbaki erreal izanik. Topologia horretan $\left\{\mathfrak{m}^N\right\}_{N\in\mathbb{N}}$ familia 0-ren ingurune oinarria da, eta hori da, hain zuzen ere, topologiaren izenaren jatorria. Are gehiago, aitzineko norma ez-arkimedearra da, hau da, *ezberdintza triangular gogorra*

$$||x+y|| \leq \max\{||x||, ||y||\} \ \forall x, y \in R$$

betetzen du, eta, beraz, R *espazio ultrametrikoa* da. Ondorioz, R osoa bada eta $\sum_{N\in\mathbb{N}} a_N$ R-ko elementuen serieak $a_N \in \mathfrak{m}^N$ betetzen badu, serie hori konbergentea da (ikusi [68, Zatia II, Kapitulua I, 64 orriko Teorema]).

Hori guztia aintzat hartuta, nagusiki eraztun hauekin eginen dugu lan:

Definizioa 1.1. *Pro-p domeinu* bat $(R, \mathfrak{m})$ integritate domeinu noetherdar lokal bat da zeina osoa den topologia $\mathfrak{m}$-adikoarekiko eta zeinaren hondar-gorputza p karakteristikako gorputz finitua den.

Oharra. Oroitu $\varprojlim_{N\in\mathbb{N}} R/\mathfrak{m}^N$ alderantzizko limitea $(R, \mathfrak{m})$ eraztun lokalaren osaketa dela topologia $\mathfrak{m}$-adikoarekiko (ikusi [32, Proposizioa 2.15]). Horrenbestez, R pro-p domeinua denean, osaketa hori R bera da, eta hondar-gorputza p^c tamainako gorputz finitua denez, R pro-p taldea da batuketarekiko.

Domeinu horien adibide aipagarrienak $\mathbb{Z}_p[[t_1, \ldots, t_m]]$ $(m \geq 0)$ eta $\mathbb{F}_q[t_1, \ldots, t_m]]$ $(m \geq 1)$ dira. Hots, $\mathbb{Z}_p$ zenbaki oso p-adikoak, koefizienteak $\mathbb{Z}_p$-n dituzten berretura serieen eraztunak eta koefizienteak $\mathbb{F}_q$-n, q elementuko gorputz finitua, dituzten berretura serieen eraztunak. Are gehiago, teorema klasiko honen arabera, zein-nahi pro-p domeinu ez dago aipatutako adibide horietatik oso urruti:

Teorema 1.2 (Cohenen Egitura Teorema [21])**.** *Izan bitez R pro-p domeinua eta $m = \dim_{\mathrm{Krull}}(R)$. Orduan, badago $S \subseteq R$ azpieraztuna zeina $S \cong \mathbb{Z}_p[[t_1, \ldots, t_{m-1}]]$ den,* $\operatorname{char} R = 0$ *denean edo $S \cong \mathbb{F}_p[[t_1, \ldots, t_m]]$ den* $\operatorname{char} R = p$ *positiboa denean. Orobat, R finituki sortutako S-modulua da.*

Bereziki, $\mathbb{Z}_p$ eta $\mathbb{F}_p[[t]]$ ideal nagusietako domeinuak (IND) eta R integritate domeinua direnez:

Korolarioa 1.3. *Izan bedi R bat Krull dimentsioko pro-p domeinua. Orduan, R finituki sortutako $\mathbb{Z}_p$-modulu askea da* $\operatorname{char} R = 0$ *bada, edo finituki sortutako $\mathbb{F}_p[[t]]$-modulu askea da* $\operatorname{char}(R) = p$ *positiboa bada.*

Hemendik aurrera, R pro-p domeinu bat izanen da eta $\mathfrak{m}$ haren ideal maximala. Halaber, besterik ezean, ageri diren kontzeptu topologiko guztiek topologia $\mathfrak{m}$-adikoari eginen diote erreferentzia. Gainera, K erabiliko dugu $\operatorname{Frac} R$, R-ren *zatikien gorputza*, adierazteko.

Aitzineko norma $K = \mathrm{Frac}(R)$ zatikien gorputzera hedatu daiteke. Izan ere, definitu

$$\left|\left|\frac{x}{y}\right|\right|_K := \frac{||x||_R}{||y||_R},$$

$x \in R$ eta $y \in R \setminus \{0\}$ guztietarako. Hala ere, R INDa izan ezean, $||\cdot||_K$ normak definituriko topologiak ez du R-ren topologia $\mathfrak{m}$-adikoa hedatzen, $\mathfrak{m}^N$ ez baita K-ren azpimultzo irekia lehenengoarekiko.

Aitzitik, $(R, \mathfrak{m})$ INDa bada, $k + \mathfrak{m}^N$ multzoa, non $k \in K$ eta $N \in \mathbb{N}$ diren, irekia da K-ren topologia naturalarekiko. Horrela, π eraztunaren *uniformizatzailea* bada, hau da, $\mathfrak{m}$ ideal maximalaren sortzaile bat, $k \in K$ elementu bakoitzerako badago $N \in \mathbb{N}_0$ zenbaki osoa non $\pi^N k \in R \backslash \mathfrak{m}$ den, eta, horrenbestez, K-ko norma modu baliokide honetan defini daiteke:

$$||k||_K = c^N$$

(definizio hau ez da uniformizatzailearen menpekoa; izan ere, π eta ρ bi uniformizatzaile badira, existitzen da $u \in \mathcal{U}(R)$ unitatea non $\rho = u\pi$ den).

Ikusiko denez, bereizketa hau, IND diren eta IND ez diren pro-p domeinuen artean, funtsezkoa da barietate analitikoen teorian.

1.1.1 Berretura serie eraztunak pro-p domeinuen gainean

Deitu $Q[[X_1, \ldots, X_n]]$ koefizienteak Q-n dituzten n aldagaitako berretura serie formalen eraztunari. Hots,

$$\mathbf{F}(X_1, \ldots, X_n) = \sum_{\alpha_i \in \mathbb{N}_0} a_{\alpha_1, \ldots, \alpha_n} X_1^{\alpha_1} \ldots X_n^{\alpha_n},$$

$a_{\alpha_1, \ldots, \alpha_n} \in Q$ izanik, elementuek osatzen duten eraztuna. Finka ditzagun tesian zehar erabiliko ditugun laburdura batzuk: $a_\alpha = a_{\alpha_1, \ldots, \alpha_n}$, $\mathbf{X}^\alpha = X_1^{\alpha_1} \ldots X_n^{\alpha_n}$, $F(\mathbf{X}) = F(X_1, \ldots, X_n)$ eta $Q[[\mathbf{X}]] = Q[[X_1, \ldots, X_n]]$. Baita $|\alpha| = \sum_{i=1}^n \alpha_i \in \mathbb{N}_0$ graduazioa ere edozein $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^{(n)}$ tuplarentzat.

Pro-p domeinuak eraztun topologikoak direnez, berretura serieen konbergentzia azter daiteke. Alegia, $F(\mathbf{X}) \in K[[\mathbf{X}]]$ koefizienteak $K = \mathrm{Frac}(R)$ zatikien gorputzean dituen berretura seriea konbergentea da $\mathbf{x} = (x_1, \ldots, x_n) \in R^{(n)}$ puntuan, $\sum_\alpha a_\alpha \mathbf{x}^\alpha$ R-ko elementuen seriea konbergentea bada. Horrela, besterik gabe $F(\mathbf{x})$ idazterakoan inplizituki F berretura serie formala $\mathbf{x}$ puntuan konbergentea dela adierazten dugu.

Lema 1.4. *Izan bitez R pro-p domeinua eta $F(X_1,\dots,X_n) \in R[[X_1,\dots,X_n]]$ berretura serie formala. Orduan, $\boldsymbol{x} \in \mathfrak{m}^{(n)}$ bada, F konbergentea da $\boldsymbol{x}$ puntuan.*

Orokorrago, izan bedi R pro-p domeinua, eta laburtzeko idatz dezagun $\Lambda = \mathrm{Frac}(R)$ R INDa bada eta $\Lambda = R$ bestelakoan. Horrela, [24, Lema 6.45] dela eta, 0-ren $\left(\mathfrak{m}^M\right)^{(n)}$ motako ingurune ireki batean konbergenteak diren $\Lambda[[\mathbf{X}]]$-ko berretura serieen multzoa

$$\Lambda_0[[\mathbf{X}]] := \left\{ \sum_{\alpha \in \mathbb{N}_0^{(n)}} a_\alpha \mathbf{X}^\alpha \in \Lambda[[\mathbf{X}]] \,\middle|\, \exists N \in \mathbb{N} \text{ non } a_\alpha \mathfrak{m}^{N|\alpha|} \subseteq R \ \forall \alpha \neq \mathbf{0} \right\} \quad (1.1)$$

da. Ohartu R ez denean INDa $\Lambda_0[[\mathbf{X}]] = R[[\mathbf{X}]]$ baino ez dela.

Berretura serie horien zenbait propietate ezagun gogoratuko ditugu, tesian zehar behin eta berriz erabiliko baitira. Irakurleak [10, Kapitulua III], [24, Atala 6.6] edo [68, Zatia II, Kapitulua II] kontsulta ditzake gaian sakondu nahi izanez gero.

Lema 1.5 (cf. [24, Lema 13.3])**.** *Demagun $F \in \Lambda_0[[X_1,\dots,X_n]]$ berretura serie formala konbergentea dela $\left(\mathfrak{m}^N\right)^{(n)}$-n. Orduan, $F\colon \left(\mathfrak{m}^N\right)^{(n)} \to R$ ebaluaketa funtzioa jarraitua da.*

Teorema 1.6 (Berretura serie eraztunen propietate unibertsala, cf. [10, Kapitulua III, §5, Proposizioa 6])**.** *Izan bedi $\varphi\colon Q \to P$ eraztun topologiko normatuen arteko eraztun homomorfismoa. Demagun P osoa eta linealki topologizatua dela eta izan bitez $p_1,\dots,p_n \in P$ elementu topologikoki nilpotenteak, hau da, $\lim_{j\to\infty} ||p_i^j||_P = 0$ betetzen da $i \in \{1,\dots,n\}$ guztietarako. Orduan, existitzen da*

$$\Phi_{\varphi,p_1,\dots,p_n}\colon Q[[X_1,\dots,X_n]] \to P$$

eraztun homomorfismoa non $\Phi_{\varphi,p_1,\dots,p_n}(X_i) = p_i$ den $i \in \{1,\dots,n\}$ guztietarako eta $\Phi_{\varphi,p_1,\dots,p_n}(q) = \varphi(q)$ den $q \in Q$ guztietarako. Halaber, $\varphi\colon Q \to P$ pro-p domeinuen arteko eraztun homomorfismo jarraitua bada, $\Phi_{\varphi,p_1,\dots,p_n}$ ere pro-p-domeinuen arteko homomorfismo jarraitua da.

Ohartu $(R,\mathfrak{m})$ pro-p domeinua bada, normaren definizioagatik edozein $x \in \mathfrak{m}$ elementu topologikoki nilpotentea dela.

Lema 1.7 (cf. [24, Korolarioa 6.48])**.** *Izan bitez $\boldsymbol{F} \in R[[X_1,\dots,X_n]]^{(m)}$ eta $\mathbf{G} \in R[[X_1,\dots,X_m]]^{(l)}$. Demagun $\boldsymbol{F}(\mathbf{0}) \in \mathfrak{m}^{(m)}$ dela, orduan $\mathbf{G} \circ \boldsymbol{F}$ konposaketa ongi definitua dago eta $R[[X_1,\dots,X_n]]^{(l)}$-ko elementua da.*

Froga. Izan bitez $\mathbf{G} = (G_1, \dots, G_l)$ eta $\mathbf{Y}$ indeterminatuen m-tupla bat. Ohartu $\mathbf{G} \circ \mathbf{F} = (\Phi(G_1(\mathbf{Y})), \dots, \Phi(G_l(\mathbf{Y})))$ dela, $\Phi \colon R[[Y_1, \dots, Y_m]] \to R[[X_1, \dots, X_n]]$ eraztun homomorfismoa propietate unibertsalaren bidez esleipen hauek definitzen dutelarik: $\Phi(r) = r$ da $r \in R$ guztietarako, eta $\Phi(Y_i) = F_i(X_1, \dots, X_n) \in (\mathfrak{m}, X_1, \dots, X_n)$, $i \in \{1, \dots, m\}$ guztietarako. □

Azkenik, emaitza hau funtsezkoa izanen da berretura serieekin lan egiterakoan:

Lema 1.8 (cf. [42, Lema 9] eta [68, 68. orriko Lema])**.** *Izan bitez $(R, \mathfrak{m})$ pro-p domeinua, $U \subseteq_o \mathfrak{m}^{(n)}$ azpimultzo irekia eta $F \in \Lambda_0[[X_1, \dots, X_n]]$ berretura serie formala. Demagun $F(\boldsymbol{x}) = 0$ dela $\boldsymbol{x} \in U$ guztietarako. Orduan, $F = 0$ da.*

Froga. Aldagai kopuruan indukzioz demagun $n = 1$ dela, eta, beraz, $F(X) = \sum_{i \geq 0} a_i X^i$ dela.

Demagun lehenengo $F(X) \in R[[X]]$ eta $0 \in U$ direla. Absurdura eramanez, suposa dezagun $F(X) \neq 0$ dela eta hartu $m \in \mathbb{N}_0$, $a_m \neq 0$ betetzen duen zenbaki arrunt txikiena. Definitu $G(X) = \sum_{i \geq m} a_i X^{i-m}$ berretura seriea, orduan $G(0) = a_m \neq 0$ da, eta jarraitutasuna dela eta badago $V \subseteq_o U$ 0-ren ingurune irekia non G ez den anulatzen. Are gehiago, R nilpotente-gabea denez, X^m ez da $V \setminus \{0\}$-n anulatzen. Horrenbestez, R integritatea domeinua denez,

$$F(X) = X^m \sum_{i \geq m} a_i X^{i-m} = X^m G(X)$$

ez da $V \setminus \{0\} \subseteq U$ azpimultzo ez-hutsean anulatzen, hasierako hipotesiarekin kontraesana dena.

Bigarrenik, demagun R INDa eta $F(X) \in K[[X]]$ direla, baina $0 \in U$ dela. Hartu π uniformizatzailea, orduan, badago $N \in \mathbb{N}$ zenbaki osoa non $\mathfrak{m}^N \subseteq U$ eta

$$\overline{a}_i := a_i \pi^{iN} \in R$$

diren $i \geq 0$ guztietarako. Horrela, hartu $\overline{F}(X) := \sum_{i \geq 0} \overline{a}_i X^i \in R[[X]]$. Serie hori $\mathfrak{m}^N$-n konbergentea denez eta bertan zero balioa hartzen duenez, lehenengo kasuaren arabera $\overline{F}(X) = 0$ dugu, hau da, $a_i \pi^{iN} = 0$ da $i \geq 0$ guztietarako. Beraz, R integritate domeinua denez, $a_i = 0$ da $i \geq 0$ guztietarako.

Azkenik, izan bedi $\Lambda = K$ R INDa bada eta $\Lambda = R$ bestelakoan. Hartu $x \in U$ eta propietate unibertsala erabilita definitu $\Phi \colon \Lambda[[X]] \to \Lambda[[X]]$ eraztun isomorfismoa $X \mapsto X + x$ esleipenaren bidez. Horrela, $\Phi(F)$ berretura serieak $-x + U := \{-x + u \mid u \in U\}$-n zero balioa hartzen du eta multzo horrek 0 barruan du. Beraz, aurreko kasuen arabera $\Phi(F) = 0$ da, eta $F = 0$ da, Φ eraztun isomorfismoa baita. □

1.2 Talde R-analitikoak

Atal honetan tesiaren aztergaia aurketuzko dugu, talde R-analitikoak hain zuzen ere.

Definizioa 1.9. Izan bedi $U \subseteq_o R^{(n)}$ azpimultzo irekia. Orduan, $f\colon U \to R$ funtzioa $x \in U$-n *R-analitikoa* dela diogu existitzen badira $N \in \mathbb{N}$ eta $F \in \Lambda_0[[X_1, \ldots, X_n]]$ berretura serie formal konbergentea non

(i) $x + \left(\mathfrak{m}^N\right)^{(n)} \subseteq U$ eta

(ii) $f(x+y) = F(y)$ den $y \in \left(\mathfrak{m}^N\right)^{(n)}$ guztietarako.

Era berean, $f = (f_1, \ldots, f_m)\colon U \to R^{(m)}$ funtzioa x puntuan R-analitikoa da, f_i guztiak x-n R -analitikoak badira, eta $f = (f_1, \ldots, f_m)\colon U \to R^{(m)}$ funtzioa U-n R-analitikoa da U-ko puntu guztietan R-analitikoa bada.

Ohartu f funtzioa puntu batean analitikoa bada puntuaren ingurune ireki batean ere hala dela. Izan ere, definizioko notazioarekin, f funtzioa x puntuan analitikoa bada, $z \in x + \left(\mathfrak{m}^N\right)$-n analitikoa da. Izan ere,

$$f(z+y) = f(x+y+(z-x)) = F(y+(z-x)) \; \forall y \in \left(\mathfrak{m}^N\right)^{(d)}$$

da eta $F(X+(z-x))$ berretura serie konbergentea da; azken batean $\Phi(F)$ baino ez da, Φ eraztun isomorfismoa propietate unibertsalaren bidez $X \mapsto X + (z-x)$ esleipenak definitzen duelarik.

Definizioa 1.10. Izan bitez R pro-p domeinua eta M espazio topologikoa.

(i) M-ren *R-karta* bat (U, ϕ, n) hirukote bat da, non $U \subseteq_o M$, $n \in \mathbb{N}_0$ eta $\phi\colon U \to \phi(U) \subseteq R^{(n)}$ homeomorfismoa diren, $\phi(U)$-ri $R^{(n)}$-ko azpiespazio topologia jarriz .

(ii) M-ren *atlas* bat $\mathcal{A} = \{(U_i, \phi_i, n_i)\}_{i \in I}$ R-karta bilduma bat da non $M = \cup_{i \in I} U_i$ den eta karta guztiak *binaka bateragarriak* diren, hau da,

$$\phi_i \circ \phi_j^{-1}|_{\phi_j(U_i \cap U_j)}\colon \phi_j(U_i \cap U_j) \to R^{(n_i)}$$

koordenatu aldaketa funtzioak R-analitikoak dira $i, j \in I$ guztietarako.

Bi atlas, $\mathcal{A}$ eta $\mathcal{B}$, *bateragarriak* dira $\mathcal{A} \cup \mathcal{B}$ ere M-ren atlas bat denean, eta atlas bat *maximala* dela diogu beste zein-nahi atlas bateragarri haren parte denean.

(iii) *Barietate R-analitiko* bat atlas maximal batek definitutako egitura analitikoaz hornitutako espazio topologikoa da.

Tesian zehar erabiliko diren zenbait termino zehaztea komeni da:

Oharra 1.11. (i) Karta bat, (U, ϕ, n), *x puntuarena* dela diogu $x \in U$ dela adierazteko.
(ii) Karta bat, (U, ϕ, n), *erregularra* da $\phi(U) = x + \left(\mathfrak{m}^N\right)^{(n)}$ bada $x \in R^{(n)}$ eta $N \in \mathbb{N}$ batentzat. Hori murrizketa tekniko bat baino ez da, azpimultzo txikiago batera pasata edozein punturen inguruan R-karta bateragarri erregular bat eraiki baitezakegu.
(iii) Karta baten, (U, ϕ, n), n zenbaki osoa kartaren *dimentsioa* da. Horrela, x puntuaren *dimentsio analitikoa*, $\dim_x M$ adieraziko duguna, x-ren edozein kartaren dimentsioa da. Teoria garatu ahala, Korolarioa 1.30en frogatuko dugu (U_1, ϕ_1, n_1) eta (U_2, ϕ_2, n_2) kartek $U_1 \cap U_2 \neq \varnothing$ betetzen badute, orduan $n_1 = n_2$ dela. Bereziki, dimentsio analitikoa ongi definituta dago, ez baita karten menpekoa.

Aipatu behar da pro-p domeinuen gainean, ohiko barietate erreal edo konplexuetan ez bezala, dimentsioa ez dela propietate topologikoa, kartak zehazten duen propietate hertsiki analitikoa baizik. Adibidez, $\mathbb{Z}_p$ eta $\mathbb{Z}_p^{(2)}$ talde topologiko isomorfoak dira.
(iv) Barietate R-analitiko bat *purua* da $\dim_x M$ konstantea bada $x \in M$ guztietarako.

Definizioa 1.9ren antzeko eran, barietate R-analitikoen arteko funtzio R-analitikoak defini daitezke:

Definizioa 1.12. Izan bitez M eta N barietate R-analitikoak. Orduan, $F\colon N \to M$ *funtzio R-analitikoa* da $x \in N$ puntuan existitzen badira x-ren (U, ϕ, n) R-karta N-n eta $F(x)$-ren (V, ψ, m) R-karta M-n non $F^{-1}(V)$ irekia den U-n eta

$$\psi \circ F \circ \phi^{-1}|_{\phi(U \cap F^{-1}(V))}\colon \phi\left(U \cap F^{-1}(V)\right) \to R^{(m)} \tag{1.2}$$

funtzio R-analitikoa den (Definizioa 1.9(i)eko zentzuan). Era berean, *F funtzio R-analitikoa* da N-ko puntu guztietan analitikoa denean.

Halaber, ohikoa da (1.2) funtzioari *F koordenatutan* deitzea, eta gehienetan terminologia hori informalki erabiliko dugu kartak zeintzuk diren zehaztu gabe. Bestetik, F funtzioa $S \subseteq N$ multzoan *hertsiki analitikoa* dela diogu, Definizioa 1.12n ageri diren kartek $S \subseteq U \cap F^{-1}(V)$ betetzen badute eta existitzen bada $\mathbf{U} \in R[[X_1, \ldots, X_n]]^{(m)}$ berretura serie formalen tupla non

$$\psi \circ F \circ \phi^{-1}(\phi(x)) = \mathbf{U}(x)$$

den $x \in S$ guztietarako. Alegia, sinpleki esatearren, funtzio hertsiki R-analitikoak berretura serie tupla *bakarraren* bidez eman daitezkeen funtzio R-analitikoak dira.

Serrek [68] erabilitako nozio hau ere baliagarria izanen da:

Definizioa 1.13. Izan bitez M barietate R-analitikoa, $U \subseteq_o M$ eta $x \in U$. Orduan, $\mathcal{F} = \{f_i \colon U \to R\}_{i=1}^{n}$ funtzio R-analitiko familia x puntuan M-ren *koordenatu sistema* da, existitzen bada $U' \subseteq_o U$ x-ren ingurune irekia non $(U', F|_{U'}, n)$, $F = (f_1, \ldots, f_n)$, R-karta bat den.

Ohartu definizioz, $\mathcal{F}$ funtzio familia x puntuan koordenatu sistema bada, lokalki x-ren inguruan ere hala dela.

Adibideak 1.14. (i) Barietate R-analitikoen adibide kanonikoa $M = \left(\mathfrak{m}^N\right)^{(n)}$ da, non $N, n \in \mathbb{N}$ den. Horrela, *koordenatu sistema kanonikoa* $\{\pi_i\}_{i=1}^{n}$ baino ez da, $\pi_i \colon \left(\mathfrak{m}^N\right)^{(n)} \to \mathfrak{m}^N$ *i*garren proiekzio funtzioa izanik. Bestetik, edozein $x \in M$-rako, $t_x \colon M \to M$, $y \mapsto y + x$ *translazio funtzioaren* osagaiek $\{\pi_i \circ t_x\}_{i=1}^{n}$ koordenatu sistema bat osatzen dute.

(ii) Izan bedi $K = \mathrm{Frac}(R)$ eta hornitu $K^{(n)}$ espazioa $k \in K^{(n)}$ puntuaren $\left\{k + \left(\mathfrak{m}^N\right)^{(n)}\right\}_{N \in \mathbb{N}}$ ingurune oinarriak definituriko topologiarekin. Orduan, $K^{(n)}$ barietate R-analitiko purua da $\{(U_k, \varphi_k, n)\}_{k \in K^{(n)}}$ atlasarekin, non $U_k = k + R^{(n)}$ eta $\varphi_k \colon U_k \to R^{(n)}$, $x \mapsto x - k$ diren (konturatu adibide honetan $K^{(n)}$-ri inposatzen ari gatzaizkion topologia ez dela oro har zatidura gorputzaren topologia naturala. Are gehiago, R eraztuna INDa izan ezean bi topologiak ez datoz bat).

(iii) Izan bitez $\mathrm{M}_{n \times m}(\mathfrak{m})$ matrizeak. $\mathrm{M}_{n \times m}(\mathfrak{m})$ multzoa naturalki $\mathfrak{m}^{(nm)}$-rekin identifika dezakegu, eta, beraz, barietate R-analitikoa da. Era berean, *talde lineal orokorra*, $\mathrm{GL}_n(R)$, barietate R-analitikoa da $\{(U_A, \phi_A, n^2)\}_{A \in \mathrm{GL}_n(R)}$ atlasarekin, non $U_A = A + \mathrm{M}_n(\mathfrak{m})$ eta $\phi_A \colon U_A \to \mathrm{M}_n(\mathfrak{m})$, $A + M \mapsto M$ diren.

(iv) Izan bitez M eta N barietate R-analitikoak hurrenez hurren $\{(U_i, \phi_i, n_i)\}_{i \in I}$ eta $\{(V_j, \psi_j, m_j)\}_{j \in J}$ atlasekin. Orduan, $M \times N$ biderketa ere barietate R-analitikoa da $\{(U_i \times V_j, \phi_i \times \psi_j, n_i + m_j)\}_{i \in I,\ j \in J}$ atlasarekin.

Propietate hauek erraz ondorioztatzen dira Lema 1.5 eta Lema 1.7tik:

Lema 1.15. (i) *Funtzio R-analitiko guztiak jarraituak dira.*

(ii) (cf. [24, Lema 13.4]) *Funtzio R-analitikoen konposaketa R-analitikoa da.*

(iii) *Funtzio hertsiki R-analitikoen konposaketa hertsiki R-analitikoa da.*

Bukatzeko, osagai guztiak bilduz, eman dezagun tesiko definizio nagusia:

Definizioa 1.16. *Talde R-analitiko* bat barietate analitikoa ere baden G talde topologiko bat da,

(i) $m\colon G \times G \to G$, $(g, h) \mapsto g \cdot h$ *biderketa funtzioa* eta

(ii) $\iota\colon G \to G$, $g \mapsto g^{-1}$ *alderantzizko funtzioa*

funtzio R-analitikoak izanik.

Talde $\mathbb{Z}_p$-analitikoak aldi berean definizio horren jatorria eta adibide nagusia dira, eta literaturan besterik gabe *talde p-adiko analitiko* deitu ohi zaie.

1.3 Talde R-estandarrak

Talde R-estandarrak talde R-analitikoen familia berezia dira.

Definizioa 1.17. Izan bedi S talde R-analitikoa. Orduan, S taldea N mailako eta d dimentsioko *talde R-estandarra* dela diogu $\{(S, \phi, d)\}$ karta globala existitzen denean non honakoa betetzen den:

(i) $\phi(S) = \left(\mathfrak{m}^N\right)^{(d)}$,

(ii) $\phi(1) = \mathbf{0}$ eta

(iii) $j \in \{1, \ldots, d\}$ guztietarako badaude $F_j \in R[[X_1, \ldots, X_{2d}]]$ berretura serie formalak non

$$\phi(xy) = (F_1(\phi(x), \phi(y)), \ldots, F_d(\phi(x), \phi(y))) \;\; \forall x, y \in S.$$

Aurreko definizioko (iii) baldintza betetzen duen edozein $\mathbf{F} = (F_1, \ldots, F_d)$ berretura serie tuplak honako bi propietate hauek ere betetzen ditu:

(F1) $\mathbf{F}(\mathbf{X}, \mathbf{0}) = \mathbf{X}$ eta $\mathbf{F}(\mathbf{0}, \mathbf{Y}) = \mathbf{Y}$ (bereziki, F_i guztiek zero gai askea dute), eta

(F2) $\mathbf{F}(\mathbf{X}, \mathbf{F}(\mathbf{Y}, \mathbf{Z})) = \mathbf{F}(\mathbf{F}(\mathbf{X}, \mathbf{Y}), \mathbf{Z})$, (hemen $\mathbf{X}$, $\mathbf{Y}$ eta $\mathbf{Z}$ aldagaien d-tuplak dira); azken batean, $\phi(1) = \mathbf{0}$ izatearen eta talde eragiketaren elkarkortasunaren berehalako ondorioak dira (ikusi [51, Kapitulua 5]). Alderantziz, $\mathbf{F} \in R[[X_1, \ldots, X_{2d}]]^{(d)}$ tupla batek aitzineko bi baldintzak betetzen baditu, $\left(\mathfrak{m}^N\right)^{(d)}$ $(N \in \mathbb{N})$ multzoa talde egitura batez hornitzen du. Hortaz, (F1) eta (F2) baldintzak betetzen dituen edozein tupla d dimentsioko *talde eragiketa formala* dela diogu. Areago, haren *alderantzizko formala* existitzen da, alegia $\mathbf{I} = (I_1, \ldots, I_d) \in R[[X_1, \ldots, X_d]]^{(d)}$ berretura serie tupla zeinak

$$\mathbf{F}(\mathbf{I}(\mathbf{X}), \mathbf{X}) = \mathbf{F}(\mathbf{X}, \mathbf{I}(\mathbf{X})) = \mathbf{0}$$

betetzen duen (ikusi [24, Proposizioa 13.16 (ii)]).

Zenbaitetan talde R-estandarrak (S, ϕ) edo $(S, \mathbf{F})$ moduan adieraziko dira, homeomorfismoa edota talde eragiketa formala zein den azpimarratzeko.

Bestalde, (F1)etik ondoriozta dezakegu

$$\mathbf{F}(\mathbf{X}, \mathbf{Y}) = \mathbf{X} + \mathbf{Y} + \mathbf{B}(\mathbf{X}, \mathbf{Y}) + \mathbf{G}(\mathbf{X}, \mathbf{Y}) \tag{1.3}$$

formakoa dela, non $\mathbf{B}$ forma bilineala den eta $\mathbf{G}$ osatzen duten monomio guztiek gutxienez 3 maila totala duten. Halaber, $\mathbf{B}$ eta $\mathbf{G}$-n ageri diren monomio guztiek X_i eta Y_j-ren berretura ez-nulu bat dute $i, j \in \{1, \ldots, d\}$ batzuetarako.

Oharra 1.18 (cf. [68, Zatia II, Kapitulua IV, §7])**.** Bestetik, (1.3) identitatetik abiatuz, alderantzizko formalarentzat eta konjokazio funtzioentzat antzeko formulak erdietsi ditzakegu. Izan ere, $\mathbf{F}$-ren alderantzizko formala $\mathbf{I}$ bada, orduan

$$\mathbf{I}(\mathbf{X}) = -\mathbf{X} + \mathbf{B}(\mathbf{X}, \mathbf{X}) + \tilde{\mathbf{G}}(\mathbf{X}) \tag{1.4}$$

da, non $\tilde{\mathbf{G}}$-n ageri diren monomio guztiek gutxienez 3 maila duten. Horrenbestez, konjokazio funtzioek forma hau dute:

$$\mathbf{F}(\mathbf{I}(\mathbf{Y}), \mathbf{F}(\mathbf{X}, \mathbf{Y})) = \mathbf{X} + \mathbf{B}(\mathbf{X}, \mathbf{Y}) - \mathbf{B}(\mathbf{Y}, \mathbf{X}) + \hat{\mathbf{G}}(\mathbf{X}, \mathbf{Y}) \tag{1.5}$$

non $\hat{\mathbf{G}}$-n ageri diren monomio guztiek gutxienez 3 maila duten eta guztietan X_i eta Y_j-ren berretura ez-nulu bat ageri den $i, j \in \{1, \ldots, d\}$ batzuetarako.

Adibideak 1.19. (i) Talde R-estandar batukorra $\left(\mathfrak{m}^N\right)^{(d)}$ multzoa da ohiko batuketarekin, hau da, $\mathbf{F}(\mathbf{X}, \mathbf{Y}) = \mathbf{X} + \mathbf{Y}$ *talde eragiketa formal batukorrarekin.*

(ii) Talde R-estandar biderkakorra $G = 1 + \mathfrak{m}^N$ da. Haren atlas globala $\phi \colon G \to \mathfrak{m}^N$, $1 + m \mapsto m$ da, eta $F(X, Y) = X + Y + XY$ *talde eragiketa formal biderkakorrak* definitzen du eragiketa.

(iii) Aurreko bi talde eragiketa formalak orokortu ditzakegu: aise frogatzen da 1 dimentsioko talde eragiketa formal polinomiko guztiek $F_c(X, Y) = X + Y + cXY$ forma dutela $c \in R$ batentzat (ikusi [9, Korolarioa 2.2.4]).

(iv) Izan bedi $\mathrm{GL}_n^1(R)$ *lehenbiziko kongruentzia taldea*: $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/\mathfrak{m})$ modulo $\mathfrak{m}$ murrizte aplikazioaren nukleoa, hau da, $\mathrm{GL}_n^1(R) = I_n + \mathrm{M}_n(\mathfrak{m})$. Talde hori talde R-estandarra da, n^2 dimentsioa du, haren R-karta globala $\mathrm{GL}^1(R) \to \mathrm{M}_n(\mathfrak{m}) = \mathfrak{m}^{n^2}$, $I_n + A \mapsto A$ da eta talde eragiketa formala $\mathbf{F}(\mathbf{X}, \mathbf{Y}) = \mathbf{X} + \mathbf{Y} + \mathbf{X} \cdot \mathbf{Y}$ da, non $\mathbf{X}$ eta $\mathbf{Y}$ n-bider-n tamainako indeterminatu matrizeak eta $\cdot$ ohiko matrize biderketa diren.

(v) Karakteristika positiboan, Chevalleyk emandako 2 dimentsioko talde eragiketa formal hau dugu:

$$\mathbf{F}(X_1, X_2, Y_1, Y_2) = (X_1 + Y_1, X_2 + Y_2 + X_1^p Y_2)$$

(ikusi [18, Kapitulua II, §10, Adibidea V]).

Talde R-estandar batek, (S, ϕ), *filtrazio serie R-estandar* hau ematen du:

$$S_n := \phi^{-1}\left(\left(\mathfrak{m}^{N+n}\right)^{(d)}\right) \ \forall n \in \mathbb{N}_0, \tag{1.6}$$

non N eta d, hurrenez hurren, taldearen maila eta dimentsioa diren. Berehalakoa da, (1.5) aintzat hartuta, $S_n \trianglelefteq_o S$ azpitalde ireki normala dela ikustea $n \in \mathbb{N}$ guztietarako. Halaber, R trinkoa denez, S talde topologiko trinkoa da, eta, beraz, S_n guztiek indize finitua dute S-n, baina are gehiago esan dezakegu:

Lema 1.20 (cf. [27, Lema 2.3]). *Izan bedi (S, ϕ, d) N mailako talde R-estandarra. Orduan,*

$$|S : S_n| = \left|\left(\mathfrak{m}^N\right)^{(d)} : \left(\mathfrak{m}^{N+n}\right)^{(d)}\right|,$$

non bigarrenak talde batukor indizea adierazten duen.

Froga. Alde batetik, (1.3) dela eta,

$$\phi(x) = \phi\left(xy^{-1}y\right) = \mathbf{F}\left(\phi\left(xy^{-1}\right), \phi(y)\right) = \phi(xy^{-1}) + \phi(y) + \mathbf{H}\left(\phi\left(xy^{-1}\right), \phi(y)\right)$$

non $\mathbf{H}(\mathbf{X}, \mathbf{Y})$-n ageri diren monomio guztiek gutxienez 2 maila duten eta X_i eta Y_j-ren berretura ez-nulu bat duten $i, j \in \{1, \ldots, d\}$ batzuetarako.

Lehenik, $\phi(x) - \phi(y) \in \left(\mathfrak{m}^K\right)^{(d)} \setminus \left(\mathfrak{m}^{K+1}\right)^{(d)}$ bada, orduan

$$\phi\left(xy^{-1}\right) + \mathbf{H}\left(\phi\left(xy^{-1}\right), \phi(y)\right) = \phi(x) - \phi(y) \in \left(\mathfrak{m}^K\right)^{(d)} \setminus \left(\mathfrak{m}^{K+1}\right)^{(d)}$$

da eta, beraz, $\phi\left(xy^{-1}\right) \in \left(\mathfrak{m}^K\right)^{(d)} \setminus \left(\mathfrak{m}^{K+1}\right)^{(d)}$. Alderantziz, $\phi\left(xy^{-1}\right) \in \left(\mathfrak{m}^K\right)^{(d)} \setminus \left(\mathfrak{m}^{K+1}\right)^{(d)}$ bada, orduan

$$\phi(x) - \phi(y) \equiv \phi\left(xy^{-1}\right) \mod \left(\mathfrak{m}^{K+1}\right)^{(d)}$$

da, eta, beraz, $\phi(x) - \phi(y) \in \left(\mathfrak{m}^K\right)^{(d)} \setminus \left(\mathfrak{m}^{K+1}\right)^{(d)}$ da.

Alegia,

$$xy^{-1} \in S_n \iff \phi(x) - \phi(y) \in \left(\mathfrak{m}^{N+n}\right)^{(d)}. \qquad \square$$

Hori dela eta, $\log_{|R/\mathfrak{m}|} |\mathfrak{m}^i : \mathfrak{m}^{i+1}| = \dim_{R/\mathfrak{m}} \left(\mathfrak{m}^i/\mathfrak{m}^{i+1}\right)$ denez,

$$|S : S_n| = p^{cd \sum_{i=0}^{n-1} \dim_{R/\mathfrak{m}}\left(\mathfrak{m}^{N+i}/\mathfrak{m}^{N+i+1}\right)} \tag{1.7}$$

da, $p^c = |R/\mathfrak{m}|$ hondar gorputzaren tamaina izanik. Bereziki, S/S_n taldea p-talde finitua da. Hortaz, kontuan hartuta S talde trinkoak identitatearen $\{S_n\}_{n\in\mathbb{N}}$ ingurune sistema bat duela non S/S_n p-talde finitua den $n \in \mathbb{N}$ guztietarako, talde R-estandarrak oinarri kontagarriko pro-p taldeak dira.

Hurrengo emaitzari esker, talde R-analitikoen azterketa zenbaitetan talde R-estandarretara murriztu daiteke.

Lema 1.21 (cf. [24, Teorema 13.20])**.** *Izan bitez G talde R-analitikoa eta (U, ϕ, d) identitatearen R-karta. Orduan, U-ren barruan $d = \dim_1 G$ dimentsioko G-ren azpitalde R-estandar ireki bat dago. Bereziki, talde R-analitiko guztiek azpitalde R-estandar ireki bat dute.*

Froga. Translazio egoki batekin konposatuz demagun $\phi(1) = \mathbf{0}$ dela. Biderketa funtzioa, m, eta alderantzizko funtzioa, ι, identitatean R-analitikoak direnez, badaude $N \in \mathbb{N}$ zenbaki osoa eta $F_j \in \Lambda_0[[X_1, \ldots, X_{2d}]]$ eta $I_j \in \Lambda_0[[X_1, \ldots, X_d]]$, $j \in \{1, \ldots, d\}$, berretura serie formalak non $\left(\mathfrak{m}^N\right)^{(d)} \subseteq \phi(U)$,

$$\phi \circ m \circ (\phi, \phi)^{-1}(x, y) = (F_1(x, y), \ldots, F_d(x, y)) \;\; \forall x, y \in \left(\mathfrak{m}^N\right)^{(d)}$$

eta

$$\phi \circ \iota \circ \phi^{-1}(x) = (I_1(x), \ldots, I_d(x)) \quad \forall x \in \left(\mathfrak{m}^N\right)^{(d)}$$

diren (egia esan, nahikoa da biderketa funtzioa bakarrik hartzea, $\left(\mathfrak{m}^N\right)^{(d)}$ irekian $\mathbf{I}$ berretura serie tupla $\mathbf{F}$-ren alderantzizko formala baino ez baita).

Bereziki, $\mathbf{F} = (F_1, \ldots, F_d)$ eta $\mathbf{I} = (I_1, \ldots, I_d)$ badira,

$$\mathbf{0} = \phi(1) = \mathbf{F}(\phi(1), \phi(1)) = \mathbf{F}(\mathbf{0}, \mathbf{0})$$

eta

$$\mathbf{0} = \phi(1) = \mathbf{I}(\phi(1)) = \mathbf{I}(\mathbf{0})$$

dira. Hots, F_j eta I_j berretura serie guztiek zero gai askea dute. Beraz, $\left(\mathfrak{m}^N\right)^{(d)}$ itxia da $\mathbf{F}$ eta $\mathbf{I}$-rekiko, alegia,

$$H := \phi^{-1}\left(\left(\mathfrak{m}^N\right)^{(d)}\right) \subseteq U$$

G-ren azpitalde irekia da eta

$$\phi(xy) = (F_1(\phi(x), \phi(y)), \ldots, F_d(\phi(x), \phi(y))) \ \ \forall x, y \in H$$

da. Horrela, R ez denean INDa, $\Lambda_0[[\mathbf{X}]] = R[[\mathbf{X}]]$ da, eta, beraz, $(H, \phi|_H)$ taldea N mailako, d dimentsioko eta $\mathbf{F}$ eragiketa lege formaldun azpitalde R-estandar irekia da.

Oinarriko R pro-p domeinua INDa denean, berriz, izan bitez π uniformizatzaile bat eta K zatikien gorputza. Orduan, (1.3)ren arabera,

$$F_j(\mathbf{X}, \mathbf{Y}) = \mathbf{X} + \mathbf{Y} + \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^{(d)} \setminus \{\mathbf{0}\} \\ |\alpha| + |\beta| \geq 2}} a_{j,\alpha,\beta} \mathbf{X}^\alpha \mathbf{Y}^\beta \in K[[\mathbf{X}, \mathbf{Y}]]$$

da, eta $\mathbf{F}$ konbergentea denez $\left(\mathfrak{m}^N\right)^{(d)}$-n, badago $L \in \mathbb{N}_0$ zenbaki osoa non $a_{j,\alpha,\beta}\pi^{L(|\alpha|+|\beta|)} \in R$ den. Definitu

$$\overline{F}_j(\mathbf{X}, \mathbf{Y}) := \pi^{-L} F_j\left(\pi^L \mathbf{X}, \pi^L \mathbf{Y}\right) = \sum_{\alpha, \beta \in \mathbb{N}_0^{(d)} \setminus \{\mathbf{0}\}} \pi^{-L} a_{j,\alpha,\beta} \pi^{L(|\alpha|+|\beta|)} \mathbf{X}^\alpha \mathbf{Y}^\beta$$

koefizienteak R-n dituen berretura seriea. Hori dela eta,

$$\bar{H} := \phi^{-1}\left(\left(\mathfrak{m}^{N+L}\right)^{(d)}\right)$$

G-ren azpitalde irekia talde R-estandarra da $\psi\colon \bar{H} \to \left(\mathfrak{m}^N\right)^{(d)}$, $h \mapsto \pi^{-L}\phi(h)$ R-karta globalarekiko eta $\overline{\mathbf{F}} = (\overline{F}_1, \ldots, \overline{F}_d)$ eragiketa lege formalarekiko. Izan ere,

$$\begin{aligned} \psi_j(xy) &= \pi^{-L}\phi_j(xy) = \pi^{-L} F_j(\phi(x), \phi(y)) \\ &= \overline{F}_j\left(\pi^{-L}\phi(x), \pi^{-L}\phi(y)\right) = \overline{F}_j\left(\psi(x), \psi(y)\right), \ \forall x, y \in H \end{aligned} \tag{1.8}$$

da. □

Talde R-estandar bat, (S, ϕ), G-ren atlas natural bat eraikitzeko erabili daiteke, S-tik eratorritako atlasa deituko duguna. Horrela, definitu $\{(xS, \phi_x, d)\}_{x \in G}$ non $\phi_x \colon xS \to \left(\mathfrak{m}^N\right)^{(d)}$, $y \mapsto \phi(x^{-1}y)$ den. Karta horiek bateragarriak dira,

$$\phi_x \circ \phi_y^{-1} = \phi \circ L_{x^{-1}} \circ L_y \circ \phi^{-1} = \phi \circ L_{x^{-1}y} \circ \phi^{-1}$$

eta $L_{x^{-1}y}$ funtzioa R-analitikoa baitira. Era berean, atlas horiek G-ren jatorrizko egitura R-analitikoarekin bateragarriak dira.

Bide batez ohartu honakoaz:

Korolarioa 1.22. *Izan bedi G talde R-analitikoa. Orduan,* $\dim_x G$ *konstantea da $x \in G$ guztietarako.*

Froga. Oharra 1.11n esan da, nahiz eta Korolarioa 1.30en frogatuko den, $\dim_x G$ R-kartekiko independentea dela. Lema 1.21en arabera, existitzen da $S \leq G$ azpitalde R-estandar irekia, $d := \dim_1 G$ dimentsiokoa, eta $\{(xS, \phi_x, d)\}_{x \in G}$ R-atlasa da, beraz, $\dim_x(G) = d$ da $x \in G$ guztietarako. □

Balio komun horri talde R-analitikoaren *dimentsio (analitikoa)* deritzo, eta, aurreko emaitza kontuan hartuta, talde R-analitikoen kartak (U, ϕ) dupla moduan idatziko ditugu.

1.3.1 Talde R-estandarrak eta talde eragiketak

Lema 1.21en arabera, talde R-analitikoek pro-p azpitalde ireki bat dute, eta, ondorioz, talde R-analitiko trinkoak talde profinituak dira. Gainera, trinkotasunaren baldintzapean Lema 1.21 gogortu daiteke:

Lema 1.23. *Demagun R pro-p domeinua ez dela INDa. Izan bedi G talde R-analitiko trinkoa. Orduan, existitzen da S azpitalde R-estandarra non $g \in G$ guztietarako $c_g \colon S \to S$ konjokazio funtzioa hertsiki R-analitikoa den.*

Froga. Izan bedi d talde R-analitikoaren dimentsioa. Lema 1.21en arabera, existitzen da indize finituko (H, ϕ) azpitalde R-estandarra. Izan bitez N H-ren maila, $\mathbf{F}$ talde eragiketa formala eta $\mathbf{I}$ alderantzizko formala. Hartu T ezker transbertsal bat H-rentzat G-n. Konjokazio funtzioak identitatean R-analitikoak direnez, $t \in T$ bakoitzerako badaude $N_t \geq N$ zenbaki osoa eta $C_j^t \in \Lambda_0[[X_1, \ldots, X_d]]$, $j \in \{1, \ldots, d\}$, berretura serieak non

$$\phi\left(x^t\right) = \left(C_1^t(\phi(x)), \ldots, C_d^t(\phi(x))\right) \ \forall x \in \phi^{-1}\left(\left(\mathfrak{m}^{N_t}\right)^{(d)}\right)$$

den. Izan bedi $L = \max_{t\in T} N_t$, orduan $\left(\mathfrak{m}^L\right)^{(d)}$ multzo irekia $\mathbf{F}$ eta $\mathbf{I}$ berretura serie tuplekiko itxia denez, $S := \phi^{-1}\left(\left(\mathfrak{m}^L\right)^{(d)}\right)$ azpitalde R-estandar irekia da. Halaber,

$$C_j^t(\mathbf{0}) = C_j^t(\phi(1)) = 0$$

denez, C_j^t bakoitzak zero gai askea du, eta, ondorioz, S itxia da edozein $t \in T$ elementurekin konjokatzearekiko eta $c_t\colon S \to S$ hertsiki analitikoa da S-n $t \in T$ guztietarako. Bestalde, $h \in H$ eta $x \in S$ denean,

$$\phi\left(x^h\right) = \mathbf{F}(\mathbf{I}(\phi(h)), \mathbf{F}(\phi(x), \phi(h))),$$

da, eta, beraz, S itxia da h-rekin konjokatzearekiko eta $c_h\colon S \to S$ hertsiki R-analitikoa da S-n. Hortaz, $g \in G$ elementu guztiak th moduan idatzi daitezkenez, $t \in T$ eta $h \in H$ izanik, orduan, S itxia da g-rekin konjokatzearekiko –hau da, S normala da G-n– eta $c_g = c_h \circ c_t$ hertsiki R-analitikoa da S-n. □

Azpiatal hau bukatu bitartean, kontserba dezagun aitzineko frogako notazioa eta errekupera dezagun (S, ϕ) azpitaldetik eratorritako R-atlasa. Atlas horren bitartez, G-ko talde eragiketen deskribapen explizitua eman dezakegu:

Lema 1.24. *Izan bedi G talde R-analitikoa eta demagun badagoela S azpitalde R-estandar normal ireki bat non $g \in G$ guztietarako $c_g\colon S \to S$ konjokazio funtzioa hertsiki R-analitikoa den. Demagun areago S-k induzitutako atlasarekiko c_g funtzioa koordinatutan $\boldsymbol{C}_g$ berretura serieak ematen duela. Izan bitez $t, r \in G$.*

(i) *Alderantzizko funtzioa tS-n koordenatutan $\boldsymbol{C}_{t^{-1}} \circ \boldsymbol{I}$ berretura serie formalen tuplak ematen du. Hots,*

$$\phi_{t^{-1}}\left(x^{-1}\right) = \left(\boldsymbol{C}_{t^{-1}} \circ \boldsymbol{I}\right)\left(\phi_t(x)\right)\ \forall x \in tS.$$

(ii) *Biderketa funtzioa $tS \times rS$-n koordenatutan $\boldsymbol{F}(\boldsymbol{C}_r(\boldsymbol{X}), \boldsymbol{Y})$ berretura serie formalen tuplak ematen du. Hots,*

$$\phi_{tr}(xy) = \boldsymbol{F}(\boldsymbol{C}_r(\phi_t(x)), \phi_r(y))\ \forall x \in tS, y \in rS.$$

Froga. (i) Hartu $x = t\overline{x} \in tS$, orduan

$$\mathbf{C}_{t^{-1}}(\mathbf{I}(\phi_t(x))) = \mathbf{C}_{t^{-1}}\left(\phi\left(\overline{x}^{-1}\right)\right) = \phi\left(\left(\overline{x}^{-1}\right)^{t^{-1}}\right) = \phi_{t^{-1}}\left(x^{-1}\right).$$

(ii) Hartu $x = t\overline{x} \in tS$ eta $y = r\overline{y} \in rS$, orduan

$$\phi_{tr}(xy) = \phi(\overline{x}^r\overline{y}) = \mathbf{F}(\mathbf{C}_r(\phi(\overline{x})), \phi(\overline{y})) = \mathbf{F}(\mathbf{C}_r(\phi_t(x)), \phi_r(y)).$$ □

Bukatzeko, finkatu T ezker transbertsal bat S-rena G-n. Ohartu $\{(tS, \phi_t)\}_{t\in T}$ atlasa nahikoa dela talde eragiketak deskribatzeko. Izan ere, hartu $x, y \in G$ non $xS = yS$ den eta definitu $A_x^y := \phi_y \circ \phi_x^{-1} \colon \left(\mathfrak{m}^N\right)^{(d)} \to \left(\mathfrak{m}^N\right)^{(d)}$ homeomorfismo R-analitikoa. Hots, $\phi_y = A_x^y \circ \phi_x$ da, eta, beraz, Lema 1.24(i) modu honetan berridatz dezakegu:

$$\phi_r\left(x^{-1}\right) = \left(A_{t^{-1}}^r \circ \mathbf{C}_{t^{-1}} \circ \mathbf{I}\right)\left(\phi_t(x)\right) \; \forall x \in tS,$$

$rS = t^{-1}S$ izanik. Berdin biderketa funtziorako, hau da,

$$\phi_p(xy) = A_{tr}^p\left(\mathbf{F}(\mathbf{C}_r(\phi_t(x)), \phi_r(y))\right) \;\; \forall x \in tS, y \in rS,$$

$trS = pS$ izanik.

1.4 Lieren aljebra

Zein-nahi talde R-estandarri Lieren aljebra bat eslei dakioke. Atal honen xedea, [24, Atala 13.3] jarraituz eraikuntza hori deskribatzea da, eta pro-p domeinu orokorretarako [68, Zatia II, Kapitulua III, § 10]eko emaitzak garatzea.

Izan bedi $(S, \mathbf{F})$ talde R-estandarra, N maila eta d dimentsiokoa. Horrela, (1.3)ko notazioarekin, $\mathbf{F}$-ri honako Lieren kortxetea elkartu dakioke

$$[\mathbf{X}, \mathbf{Y}]_{\mathbf{F}} := \mathbf{B}(\mathbf{X}, \mathbf{Y}) - \mathbf{B}(\mathbf{Y}, \mathbf{X})$$

(nahasteko arriskurik ez dagoenean sinpleki $[\cdot, \cdot]$ izendatuko dugu). Ikus dezagun $[\cdot, \cdot]_{\mathbf{F}}$ benetan Lieren kortxetea dela (ikusi Apendizea A kortxetearen definiziorako). Alde batetik, aplikazio bilineala da eta $[\mathbf{X}, \mathbf{X}] = 0$ betetzen du. Bestalde, Jacobiren identitatea betetzen du:

Lema 1.25 (cf. [24, Lema 13.24] eta [68, Zatia II, Kapitulua IV, § 7.6])**.** *Izan bitez $\boldsymbol{X}$, $\boldsymbol{Y}$ eta $\mathbf{Z}$ indeterminatuen d-tuplak. Orduan,*

$$[\boldsymbol{X}, [\boldsymbol{Y}, \mathbf{Z}]] + [\boldsymbol{Y}, [\mathbf{Z}, \boldsymbol{X}]] + [\mathbf{Z}, [\boldsymbol{X}, \boldsymbol{Y}]] = \mathbf{0}$$

da.

Froga. Emaitza Hall-Witten identitatearen (cf. [68, Zatia I, Proposizioa 1.1]) –Jacobiren identitatearen bertsio ez-trukakorra– ondorio zuzena da. Hots, talde guztiek identitate hau betetzen dute:

$$[x^y, [y, z]]\,[y^z, [z, x]]\,[z^x, [x, y]] = 1. \tag{1.9}$$

Hemendik aurrera $O(n)$ laburdura erabiko dugu, hau da, bi indeterminaturen d-tuplaz, $\mathbf{X}$ eta $\mathbf{Y}$, osatutako berretura serie formalak non horietan ageri diren monomio guztiek gutxienez n maila duten eta X_i eta Y_j baten berretura ez-nulu bat ageri den $i, j \in \{1, \ldots, d\}$ batzuetarako. Halaber, $\mathbf{F}(\mathbf{I}(\mathbf{Y}), \mathbf{F}(\mathbf{X}, \mathbf{Y}))$ konjokazio formala $\mathbf{X}^{\mathbf{Y}}$ moduan laburtuko dugu, eta

$$\mathbf{C}(\mathbf{X}, \mathbf{Y}) := \mathbf{F}(\mathbf{I}(\mathbf{X}), \mathbf{F}(\mathbf{I}(\mathbf{Y}), \mathbf{F}(\mathbf{X}, \mathbf{Y})))$$

erabiliko dugu konmutadore adierazteko. Horrela, (1.3)-(1.5)en arabera, $\mathbf{X}^{\mathbf{Y}} = \mathbf{X} + O(2)$ eta $[\mathbf{X}, \mathbf{Y}] = \mathbf{C}(\mathbf{X}, \mathbf{Y}) + O(3)$ dira. Hori dela eta,

$$[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] = \mathbf{C}\left(\mathbf{X}^{\mathbf{Y}}, \mathbf{C}(\mathbf{Y}, \mathbf{Z})\right) + O(4)$$

da. Hortaz, (1.9) identitatearen arabera,

$$\begin{aligned} \mathbf{0} &= \mathbf{F}\left(\mathbf{C}\left(\mathbf{X}^{\mathbf{Y}}, \mathbf{C}\left(\mathbf{Y}, \mathbf{Z}\right)\right), \mathbf{F}\left(\mathbf{C}\left(\mathbf{Y}^{\mathbf{Z}}, \mathbf{C}\left(\mathbf{Z}, \mathbf{X}\right)\right), \mathbf{C}\left(\mathbf{Z}^{\mathbf{X}}, \mathbf{C}\left(\mathbf{X}, \mathbf{Y}\right)\right)\right)\right) \\ &= [\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] + O(4), \end{aligned}$$

Beraz, $[\cdot, \cdot]$-n ageri diren monomio guztiek 2 maila dutenez,

$$[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = \mathbf{0}. \qquad \square$$

Horrenbestez, Lieren kortxete horrek $R^{(d)}$-ren gainean *Lieren aljebra* egitura definitzen du, *S-ri elkarturiko Lieren aljebra* deituko duguna.

Bestetik, R-Lie aljebren arteko aplikazio naturalak R-Lie aljebra homomorfismoak dira, hau da, Lieren kortxeak gordetzen dituzten aplikazio R-linealak. Halaber, funtzio R-analitiko baten hurbilketa lineala haren diferentzialak ematen du. Hots, izan bedi $\mathbf{U}(\mathbf{X}) \in R[[X_1, \ldots, X_n]]^{(m)}$ berretura serie formalen tupla eta demagun $\mathbf{U}(\mathbf{0}) = \mathbf{0}$ dela. Deitu $\mathbf{U}^{[k]}$ moduan $\mathbf{U}$-ren k mailako zati homogeneoari. Bereziki, $\mathbf{U}$-ren zati lineala

$$\mathbf{U}^{[1]}(\mathbf{X}) = \left(\sum_{i=1}^{n} a_{1i} X_i, \ldots, \sum_{i=1}^{n} a_{mi} X_i\right)$$

da. Orduan, $\mathbf{U}$-ren *diferentziala* $\mathbf{U}^{[1]}$-k definitutako $D\mathbf{U}\colon R^{(n)} \to R^{(m)}$ aplikazio lineala da, eta $\mathbf{U}$-ren *matrize jacobiarra* $\mathcal{J}\mathbf{U} := (a_{ij})_{i,j}$ matrizea da, diferentziala definitzen duen matrizea hain zuzen ere. Zailtasun handirik gabeko ariketa da *katearen erregela* frogatzea, hau da, $\mathbf{V}$ berretura serieen n-tupla bada eta $\mathbf{V}(\mathbf{0}) = \mathbf{0}$ betetzen badu, orduan

$$D(\mathbf{U} \circ \mathbf{V}) = D\mathbf{U} \circ D\mathbf{V} \text{ eta } \mathcal{J}(\mathbf{U} \circ \mathbf{V}) = \mathcal{J}\mathbf{U} \cdot \mathcal{J}\mathbf{V}$$

dira.

Era berean, izan bedi $f\colon U \subseteq R^{(n)} \to R^{(m)}$ funtzio R-analitikoa eta demagun $f(x) = 0$ dela. Definizioa 1.12 dela eta, badago $\mathbf{U}$ berretura serie formala non $\mathbf{U}(y) = f(x+y)$ den $y \in U$ guztietarako, U x-ren ingurune egoki bat izanik. Bereziki, $\mathbf{U}(\mathbf{0}) = \mathbf{0}$ da. Orduan, f-ren diferentziala x-n, $D_x f$ izendatuko duguna, $D\mathbf{U}$ da, eta f-ren matrize jacobiarra x-n, $\mathcal{J}_x f$ izendatuko duguna, $\mathcal{J}\mathbf{U}$.

Bi talde R-estandarren, $(S, \mathbf{F}, d)$ eta $(T, \mathbf{G}, e)$, arteko *morfismo formal* bat $\mathbf{U} \in R[[X_1, \ldots, X_d]]^{(e)}$ berretura serie formalen tupla bat da non

$$\mathbf{U}(\mathbf{F}(\mathbf{X}, \mathbf{Y})) = \mathbf{G}(\mathbf{U}(\mathbf{X}), \mathbf{U}(\mathbf{Y}))$$

den. Ikus dezagun diferentzialak Lieren kortxeteak gordetzen dituela:

Lema 1.26 (cf. [24, Proposizioa 13.26] eta [35, Atala 14.2])**.** *Izan bedi $\boldsymbol{U}$ talde morfismo formala. Orduan, $D\boldsymbol{U}$ diferentziala R-Lie aljebra homomorfismoa da, hau da,*

$$\mathbf{U}^{[1]}\left([\boldsymbol{X}, \boldsymbol{Y}]_{\boldsymbol{F}}\right) = \left[\mathbf{U}^{[1]}(\boldsymbol{X}), \mathbf{U}^{[1]}(\boldsymbol{Y})\right]_{\mathbf{G}}.$$

Froga. Froga laburki deskribatu baino ez dugu eginen. Sinpletasunagatik erabili $\cdot$ eta $\cdot^{-1}$ biderketa eta alderantzizkoa izendatzeko, dagozkien eragiketa eta alderantzizko formalen ordez. Gainera, $O(n)$ notazioa Lema 1.25eko era berean erabiliko da. Bereziki, $\mathbf{U}(\mathbf{X}) = \mathbf{U}^{[1]}(\mathbf{X}) + O(2)$ da, eta, ondorioz,

$$\mathbf{U}(\mathbf{X})^{-1} \cdot \mathbf{U}(\mathbf{Y})^{-1} \cdot \mathbf{U}(\mathbf{X}) \cdot \mathbf{U}(\mathbf{Y}) = \left[\mathbf{U}^{[1]}(\mathbf{X}), \mathbf{U}^{[1]}(\mathbf{Y})\right]_{\mathbf{G}} + O(3)$$

eta

$$\mathbf{U}\left(\mathbf{X}^{-1} \cdot \mathbf{Y}^{-1} \cdot \mathbf{X} \cdot \mathbf{Y}\right) = \mathbf{U}^{[1]}\left([\mathbf{X}, \mathbf{Y}]_{\mathbf{F}}\right) + O(3),$$

ditugu. Bi mailako monomioak konparatuz lortzen da emaitza. □

Hortaz, talde R-estandar batek hainbat R-karta global eta horren eraginez hainbat eragiketa lege formal, izan ditzakeen arren, elkarturiko Lieren aljebra guztiak isomorfoak dira $D_{\mathbf{0}}\left(\phi_1 \circ \phi_2^{-1}\right)$-ren bidez (hemen ϕ_i R-karta globalak dira). Hots, S-ri elkarturiko Lieren aljebra, isomorfismoak salbu, bakarra da. Are gehiago, orain arteko guztiak kontuan hartuta, talde R-estandar bati Lieren aljebra bat esleitzearen ideia kategorikoki formalizatu daiteke.

Teorema 1.27 (cf. [24, Atala 13.3])**.** *Orain arteko notazioa mantenduz: $(S, \mathbf{F}) \mapsto \left(R^{(d)}, [\cdot,\cdot]_{\boldsymbol{F}}\right)$ eta $f \mapsto D_{\mathbf{0}} f$ esleipenek funktore bat definitzen dute, $\mathfrak{RStd}$, objektu gisa $(S, \boldsymbol{F})$ talde R-estandarrak eta morfismo gisa f talde morfismo formalak dituen kategoriaren, eta $\mathfrak{RLie}$, objektu gisa R-Lie aljebrak eta morfismo gisa R-Lie algebra homomorfismoak dituen kategoriaren artean.*

Izan bitez $F\colon N \to M$ funtzio R-analikoa, x-ren (U, ϕ, n) R-karta N-n eta $F(x)$-ren (V, ψ, m) R-karta M-n, eta demagun areago $\phi(x) = \mathbf{0}$ eta $\psi(F(x)) = \mathbf{0}$ direla (hau suposizio teknikoa da, translazio egoki batekin konposatuz halako homeomorfismoak lor ditzakegu eta). Orduan, F-ren *diferentziala* eta *matrize jacobiarra* hurrenez hurren $D_x F$ eta $\mathcal{J}_x F$ izendatuko ditugunak, $D_{\mathbf{0}}\left(\psi \circ F \circ \phi^{-1}\right)$ eta $\mathcal{J}_{\mathbf{0}}\left(\psi \circ F \circ \phi^{-1}\right)$ moduan definitzen dira. Definizio horiek jacobiar analitiko klasikoarekin erlazionatuta daude, hots, $\mathbf{U}(\mathbf{X}) \in R[X_1, \ldots, X_n]]^{(m)}$ bada, haren matrize jacobiar "klasikoa"

$$\operatorname{Jac}\mathbf{U}(X_1, \ldots, X_n) := \left(\partial_j U_i(X_1, \ldots, X_n)\right)_{i,j}$$

da, non ∂_j ikurrak X_j aldagaiarekiko deribatu formala adierazten duen. Hori dela eta, lokalki $\mathbf{0}$-ren inguruan $\psi \circ F \circ \phi^{-1}(z) = \mathbf{U}(z)$ bada, $\mathcal{J}_x F = \operatorname{Jac}\mathbf{U}(\mathbf{0})$ da. Gainera, $y \in U \cap F^{-1}(V)$ denean, $\overline{\phi}(x) := \phi(x) - \phi(y)$ eta $\overline{\psi}(x) := \psi(x) - \psi(F(y))$ y-ren eta $F(y)$-ren R-kartak dira, $\overline{\phi}(y) = \mathbf{0}$ eta $\overline{\psi}(F(y)) = \mathbf{0}$ izanik. Horrenbestez, katearen erregela analitikoa dela eta,

$$\begin{aligned}\mathcal{J}_y F = \mathcal{J}_{\mathbf{0}}\left(\overline{\psi} \circ F \circ \overline{\phi}^{-1}\right) &= \mathcal{J}_{\mathbf{0}}\left(\mathbf{U}(\mathbf{X} + \phi(y)) - \psi(F(y))\right) \\ &= \operatorname{Jac}\mathbf{U}(\mathbf{X} + \phi(y))|_{\mathbf{X}=\mathbf{0}} = \operatorname{Jac}\mathbf{U}(\mathbf{X})|_{\mathbf{X}=\phi(y)}. \end{aligned} \tag{1.10}$$

Argudiaketa hori guztia kontuan hartuta, eta notazio sinplifikatzearren, atala bukatu bitartean (U, ϕ, n) x-ren R-karta bada, $\phi(x) = \mathbf{0}$ dela suposatuko dugu.

Aurreko definizio horiek R-karten menpekoak diren arren, matrize jacobiarrean berean bainoago haren heinean interesatuko gara, eta besteak beste honako emaitza hau dugu:

Lema 1.28. *Izan bedi $\mathbf{U} = (U_1, \ldots, U_n) \in R[[X_1, \ldots, X_n]]^{(n)}$ zero gai askea duen berretura serie formalen tupla, eta demagun U_i-ren zati lineala $\sum_{j=1}^{n} a_{ij} X_j$ dela. Demagun areago $\mathbf{U}$ alderanzgarria dela, hau da, badago $\mathbf{V} = (V_1, \ldots, V_n) \in R[[X_1, \ldots, X_n]]^{(n)}$ berretura serie formalen tupla non*

$$V_i \circ \mathbf{U}(\boldsymbol{X}) = U_i \circ \mathbf{V}(\boldsymbol{X}) = X_i$$

den $i \in \{1, \ldots, n\}$ guztietarako. Orduan, $(a_{ij})_{i,j} \in \mathrm{GL}_n(R)$ da.

Froga. Ohartu V_i-ren zati lineala $\sum_{j=1}^{n} b_{ij} X_j$ bada, orduan

$$(\mathbf{U} \circ \mathbf{V})(\mathbf{X}) = \left(\sum_{j=1}^{n}\sum_{k=1}^{n} a_{1k} b_{kj} X_j + O(2), \ldots, \sum_{j=1}^{n}\sum_{k=1}^{n} a_{nk} b_{kj} X_j + O(2)\right) = \mathbf{X}$$

dela, non ohiko legez $O(n)$ bidez gutxienez n mailako monomioz osatutako berretura serieak adierazten ditugun. Hortaz, $(b_{ij})_{i,j}$ da $(a_{ij})_{i,j}$-ren alderantzizko matrizea. □

Adibidea 1.29. Esate baterako, G talde R-analitikoan $L_g\colon G \to G$, $x \mapsto gx$ *ezker biderketa* alderanzgarria da, eta, beraz, Lema 1.28ren arabera, $\mathcal{J}_x L_g \in \mathrm{GL}_n(R)$ da $x \in G$ guztietarako.

Korolarioa 1.30. *Izan bitez M barietate R-analitikoa eta $\{(U_i, \phi_i, d_i)\}_{i=1}^2$ R-kartak. Orduan, $U_1 \cap U_2 \neq \varnothing$ bada, $d_1 = d_2$ da.*

Froga. Hartu $x \in U_1 \cap U_2$. Definizioz, $\phi_1 \circ \phi_2^{-1}$ funtzio R-analitiko alderanzgarria da, beraz, Lema 1.28 dela eta, $\mathcal{J}_x\left(\phi_1 \circ \phi_2^{-1}\right) \in \mathrm{M}_{d_1 \times d_2}(R)$ alderanzgarria da. Bereziki $d_1 = d_2$ da. □

Korolarioa 1.31. *Izan bitez $F\colon N \to M$ funtzio R-analitikoa eta $x \in N$. Izan bitez $\{(U_i, \phi_i, n)\}_{i=1}^2$ eta $\{(V_i, \psi_i, m)\}_{i=1}^2$ hurrenez hurren x-ren R-kartak N-n eta $F(x)$-ren R-kartak M-n. Orduan,*

$$\operatorname{rk} \mathcal{J}_{\mathbf{0}}\left(\psi_1 \circ F \circ \phi_1^{-1}\right) = \operatorname{rk} \mathcal{J}_{\mathbf{0}}\left(\psi_2 \circ F \circ \phi_2^{-1}\right).$$

Froga. Ohartu $\phi_1\left(U_1 \cap U_2 \cap F^{-1}(V_1) \cap F^{-1}(V_2)\right)$-ra murriztuta,

$$\psi_1 \circ F \circ \phi_1^{-1} = \psi_1 \circ \psi_2^{-1} \circ \psi_2 \circ F \circ \phi_2^{-1} \circ \phi_2 \circ \phi_1^{-1}$$

dela, eta, beraz,

$$\mathcal{J}_{\mathbf{0}}\left(\psi_1 \circ F \circ \phi_1^{-1}\right) = \mathcal{J}_{\mathbf{0}}\left(\psi_1 \circ \psi_2^{-1}\right) \mathcal{J}_{\mathbf{0}}\left(\psi_2 \circ F \circ \phi_2^{-1}\right) \mathcal{J}_{\mathbf{0}}\left(\phi_2 \circ \phi_1^{-1}\right)$$

dela. Horrela, $\psi_1 \circ \psi_2^{-1}$ eta $\phi_1 \circ \phi_2^{-1}$ funtzio R-analitiko alderanzgarriak direnez, emaitza Lema 1.28ren ondorioa da. □

Hori dela eta, aipatu denez $\mathcal{J}_x F$ eta $D_x F$-n definizioak, matrize eta funtzio lineal gisa, anbiguoak diren arren, "$\mathcal{J}_x F \in \mathrm{GL}_n(R)$" edo "$D_x F$ *injektiboa da*" moduko baieztapenak matematikoki zuzenak dira. Bereziki, hurrengo definizioa kartez arduratu gabe enuntzia dezakegu:

Definizioa 1.32. Izan bedi $F\colon N \to M$ funtzio R-analitikoa. Orduan,

(i) F *murgilketa ahula* da x-n, $\operatorname{rk} \mathcal{J}_x F = \dim_x N$ bada.

(ii) F *azpiraketa ahula* da x-n, $\operatorname{rk} \mathcal{J}_x F = \dim_x M$ bada.

Era berean, F *murgilketa ahula* da puntu guztietan hala denean, eta berdin *azpiraketa ahuletarako.*

Ondoko teorema [24, Teorema 6.37]ren froga moldatuz frogatu daiteke.

Teorema 1.33 (Alderatzizko Funtzioaren Teorema)**.** *Izan bedi R eraztuna eta izan bitez $U_1, \dots, U_n \in R[[X_1, \dots, X_n]]$ gai askerik gabeko berretura serieak. Izan bedi $\sum_{j=1}^n a_{ij} X_j$ polinomioa U_i-ren zati lineala. Demagun $(a_{ij})_{i,j} \in \mathrm{GL}_n(R)$ dela. Orduan, existitzen dira $V_1, \dots, V_n \in R[[X_1, \dots, X_n]]$ zero gai askea duten berretura serieak non*

$$(V_i \circ \boldsymbol{U})(\boldsymbol{X}) = (U_i \circ \boldsymbol{V})(\boldsymbol{X}) = X_i$$

den $i \in \{1, \dots, n\}$ guztietarako.

Alegia, matrize jacobiarrak funtzio baten alderanzgarritasun lokalaren berri ematen du.

Definizioa 1.34. Izan bedi $F\colon N \to M$ funtzio R-analitikoa. Orduan, F funtzio *R-bianalitikoa** da $x \in N$ puntuan, $\mathcal{J}_x F \in \mathrm{GL}_n(R)$ denean. Halaber, F *R-bianalitikoa* da M-ko puntu guztietan bianalitikoa denean.

Alegia, funtzio bianalitiko kontzeptua isomorfismo nozio naturala da barietate R-analitikoetan: alderantzizko analitikoa duten homeomorfismo R-analitikoak. Nahiz eta froga definizioen ondorio zuzena izan, emaitza hau oinarrizkoa da barietate R-analitikoen testuinguruan.

Teorema 1.35. *Izan bedi $F\colon N \to M$ funtzio R-analitikoa. Orduan, F funtzioa $x \in N$ puntuan R-bianalitikoa da baldin eta soilik baldin $D_x F$ R-modulu isomorfismoa bada.*

Aurrekoa koordenatu aldaketak aztertzeko ere erabil daiteke:

Teorema 1.36 (Koordenatu aldaketak)**.** *Izan bitez M barietate R-analitikoa, $x \in M$, $n = \dim_x M$, $U \subseteq_o M$ non $x \in U$ eta $\{f_i\colon U \to R\}_{i=1}^n$ funtzio R-analitikoen familia non $f_i(x) = 0$ den i guztietarako. Hurrengoak baliokideak dira:*

(i) *$\{f_i\}_{i=1}^n$ koordenatu sistema da x-n.*

(ii) *$F = (f_1, \dots, f_n)$ funtzio R-bianalitikoa da x-n.*

(iii) *$\{D_x f_i\}$ R-modulu oinarria da $\left(R^{(n)}\right)^*$-rentzat ($R^{(n)}$-ren R-modulu duala).*

*Serrek [68]n *étale* terminoa darabil funtzio hauek izendatzeko.

Froga. (i) $\Rightarrow$ (ii). Existitzen da x-ren V ingurune irekia non $(V, F|_V, n)$ x-ren R-karta den, eta, beraz, karta horrekiko F identitatea da koordenatutan.
(ii) $\Rightarrow$ (i). Lokalki F homeomorfismo R-bianalitikoa da Teorema 1.35en poderioz, $\mathcal{J}_x F \in GL_n(R)$ baita. Beraz, existitzen da $V \subseteq_o U$ azpimultzo irekia non $F\colon V \to F(V) \subseteq R^{(n)}$ homeomorfismoa den, eta erraz ikusten da (V, F, n) R-karta M-ren jatorrizko egitura R-analitikoarekin bateragarria dela.
(ii) $\Leftrightarrow$ (iii). Izan bedi $(b_{ij})_{i,j}$ F-ren *edozein* matrize jacobiar x-n. Orduan, $\left\{\sum_{j=1}^n b_{1j}X_j, \ldots, \sum_{j=1}^n b_{nj}X_j\right\}$ aplikazio linealen familia $\left(R^{(n)}\right)^*$-ren oinarria da baldin eta soilik baldin $(b_{ij})_{i,j} \in \mathrm{GL}_n(R)$ bada. $\square$

Barietate analitikoen teoria batik bat gorputzen gainean, nagusiki $\mathbb{C}$, garatu izan da. Emaitza askok, halere, elkarturiko Lieren aljebraren (Lieren teorian *espazio ukitzaile* ere deitzen zaio) bektore espazio egituraren ondorio dira. Pro-p domeinu orokorren gainean, ordea, Lieren aljebrak R-moduluak direnez, argudio gehienak errepikatu daitezke, baina zenbait baldintza gehigarrirekin.

Lieren aljebrak R-moduluak izanik (ez espazio bektorialak), komeni da ohiko heinaren ordez *hondar-heina* kontzeptua erabiltzea. Hots, izan bitez Q eraztuna eta $A \in \mathrm{M}_{n\times m}(Q)$. Orduan, A matrizearen hondar-heina, $\operatorname{res.rk}(A)$ adieraziko duguna, ondoko baldintza betetzen duen $r \in \mathbb{N}_0$ zenbaki oso handiena da: badago $r \times r$ tamainako Δ_r minore bat A-n non $\Delta_r \in \mathcal{U}(Q)$ den. Noski, Q gorputza denean, kontzeptu hau ohiko heinaren baliokidea da, eta $Q = (R, \mathfrak{m})$ pro-p domeinua denean karakterizazio baliokide hau dugu:

$$\operatorname{res.rk}(A) = \operatorname{rk}_{R/\mathfrak{m}}\left(\overline{A}\right),$$

non $\overline{A}$ matrizea A-ren sarrerak modulo $\mathfrak{m}$ murriztuz erdiesten den matrizea den, eta $\operatorname{rk}_{R/\mathfrak{m}}$-k heina $R/\mathfrak{m}$ hondar gorputzean kalkulatzen dela esan nahi duen.

Korolarioa 1.31en heinaren ordez hondar-heina hartzen badugu, emaitza bera lortzen da –aski da froga hitzez hitz errepikatzea–, bereziki ondoko definizioa ez da anbiguoa.

Definizioa 1.37. Izan bedi $F\colon N \to M$ funtzio R-analitikoa. Orduan,

(i) F *murgilketa* da x-n, $\operatorname{res.rk} \mathcal{J}_x F = \dim_x N$ bada.

(ii) F *azpiraketa* da x-n, $\operatorname{res.rk} \mathcal{J}_x F = \dim_x M$ bada.

Era berean, *murgilketa* bat domeinuko puntu guztietan murgilketa den funtzio R-analitikoa da, eta berdin azpiraketetarako.

Lema 1.38. *Izan bedi $F\colon N\to M$ murgilketa R-analitikoa. Orduan, F koordenatutan*

$$\iota\colon R^{(n)}\to R^{(m)},\ (x_1,\dots,x_n)\mapsto\Big(x_1,\dots,x_n,0,\overset{(m-n)}{\dots},0\Big)$$

monomorfismo R-lineala da. Konkretuago, $x\in N$ guztietarako badago x-ren (U,ϕ,n) R-karta N-n eta $F(x)$-ren (V,ψ,m) R-karta M-n non hurrengo diagrama trukakorra den:

$$\begin{array}{ccc} U & \xrightarrow{\quad F\quad} & V \\ \phi\downarrow & & \downarrow\psi \\ R^{(n)} & \xrightarrow{\quad \iota\quad} & R^{(m)}. \end{array}$$

Era berean, azpiraketa bat koordenatutan

$$\pi\colon R^{(n)}\to R^{(m)},\ (x_1,\dots,x_n)\mapsto(x_1,\dots,x_m)$$

epimorfismo R-lineala da.

Froga. Emaitza murgilketetarako frogatuko dugu, azpiraketetarako berdin-berdin frogatzen baita. Lehenik eta behin, emaitza lokala denez, koordenatutan lan eginda, demagun $x=\mathbf{0}\in U=\left(\mathfrak{m}^L\right)^{(n)}\subseteq_o N$, $F(x)=\mathbf{0}\in V=\left(\mathfrak{m}^L\right)^{(m)}\subseteq_o M$ eta $F\in R[[X_1,\dots,X_n]]^{(m)}$ direla.

Bigarrenik, sinpletasunagatik, demagun $\mathcal{J}_{\mathbf{0}}F$-ren lehenbiziko n zutabeen hondar-heina n dela, hau da, $\tilde{F}=(F_1,\dots,F_n)$ bada, orduan $\operatorname{res.rk}\mathcal{J}_{\mathbf{0}}\tilde{F}=n$ da.

Izan bedi $W=\left(\mathfrak{m}^N\right)^{(m-n)}$, eta definitu $\Phi\colon N\times W\to M$, $(x,w)\mapsto F(x)+(\mathbf{0},w)$ aplikazioa. Orduan,

$$\mathcal{J}_{\mathbf{0}}\Phi=\left(\begin{array}{c|c} \mathcal{J}_{\mathbf{0}}F & \begin{matrix}\mathbf{0}\\ I_{m-n}\end{matrix}\end{array}\right)\in\mathrm{M}_m(R)$$

eta $\operatorname{res.rk}\mathcal{J}_{\mathbf{0}}\Phi=m$ dira. Beraz, Teorema 1.33ren arabera, existitzen da Φ-ren alderantzizko lokala, hau da, badaude U', V' eta W' azpimultzo irekiak, hurrenez hurren $R^{(n)}$, $R^{(m)}$ eta $R^{(m-n)}$-n, non $\Psi\colon V'\to U'\times W'$ funtzioa $\Phi|_{U'\times W'}$-ren alderantzizkoa den. Halaber, ondoko diagrama trukakorra da:

$$\begin{array}{ccc} U' & \xrightarrow{\quad F\quad} & V' \\ \iota\downarrow & & \downarrow\Psi \\ U'\times\{0\}^{(m-n)} & \xrightarrow{\Psi\circ F\circ\iota^{-1}} & U'\times W', \end{array}$$

non $\iota(x) = (x, \mathbf{0})$ den. Azkenik,

$$\Psi \circ F \colon U' \to U' \times W', \ (x_1, \ldots, x_n) \mapsto \Big(x_1, \ldots, x_n, 0, \overset{(m-n)}{\ldots}, 0\Big)$$

da, nahi genuen moduan. □

Bereziki, funtzio R-bianalitiko bat aldi berean murgilketa bat eta azpiraketa bat da, eta, ondorioz, koordenatutan identitatea gisa idatz daiteke.

Definizioa 1.39. Izan bedi $F \colon N \to M$ funtzio R-analitikoa. Orduan, F *azpimurgilketa* da $x \in N$ puntuan existitzen direnean $x \in U \subseteq_o N$, $F(x) \in V \subseteq_o M$ eta W barietate R-analitikoa non $F|_U$ azpiraketa baten eta murgilketa baten konposizioa den. Hots,

$$U \xrightarrow{\pi} W \xrightarrow{\iota} V$$

da, π azpiraketa bat eta ι murgilketa bat izanik.

Lema 1.40. *Izan bedi $F \colon N \to M$ funtzioa R-analitikoa. Hurrengoak baliokideak dira:*

(i) *F azpiraketa da x puntuan.*

(ii) *F koordenatutan, x-ren inguruan,*

$$\overline{F} \colon R^{(n)} \to R^{(m)}, \ (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0)$$

homomorfismo R-analitikoa gisan idatzi daiteke $r \leq \min\{n, m\}$ baterako, $n = \dim_x N$ eta $m = \dim_{F(x)} M$ izanik.

Froga. Froga Lema 1.38ren ondorioa da.

(i) ⇒ (ii). Definizioz, F lokalki $U \xrightarrow{\pi} W \xrightarrow{\iota} V$ da. Hartu $r = \dim_{\pi(x)} W$. Lema 1.38ren arabera, π eta ι koordenatutan

$$\overline{\pi} \colon R^{(n)} \to R^{(r)}, \ (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_r)$$

eta

$$\overline{\iota} \colon R^{(r)} \to R^{(m)}, \ (x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_r, 0, \overset{(m-r)}{\ldots}, 0)$$

dira, beraz, F koordenatutan horien konposizioa da.
(ii) ⇒ (i). Existitzen dira x-ren (U, ϕ, n) R-karta eta $F(x)$-ren (V, ψ, m) R-karta non $\psi \circ F \circ \phi^{-1} = \overline{F} = \overline{\iota} \circ \overline{\pi}$ den $\phi(U)$ irekian (hemen $\overline{\pi}$ eta $\overline{\iota}$ aurreko inplikaziokoak dira). Hortaz, $\overline{W} = \overline{\pi}\,(\phi(U))$ $R^{(r)}$-ren azpimultzo irekia da, eta hurrengo diagrama trukakorra da:

$$
\begin{array}{ccccc}
U & & \xrightarrow{\quad F \quad} & & V \\
\downarrow{\scriptstyle\phi} & & & & \uparrow{\scriptstyle\psi^{-1}} \\
\phi(U) & \xrightarrow{\overline{\pi}} & \overline{W} & \xrightarrow{\overline{\iota}} & \psi(V)
\end{array}
$$

Azkenik, $\overline{\pi} \circ \phi$ azpiraketa da eta $\psi^{-1} \circ \overline{\iota}$ murgilketa. □

Hurrengo lema [68, 86. orriko Teorema]ko argudioak errepikatuz frogatzen da. Honako oinarrizko emaitza hau beharko dugu:

Lema 1.41. *Izan bitez Q zero karakteristikako eraztuna eta $F \in Q[[X_1, \ldots, X_n]]$. Demagun azken m aldagaiekiko deribatu formalak 0 direla, hau da, $\partial_j F = 0$ da $j \in \{n-m+1, \ldots, n\}$ guztietarako. Orduan, $F \in Q[[X_1, \ldots, X_{n-m}]]$ da.*

Proposizioa 1.42. *Izan bitez R zero karakteristikako pro-p domeinua eta $F \colon N \to M$ funtzio R-analitikoa. Demagun existitzen dela $r \in \mathbb{N}_0$ zenbaki osoa non $\operatorname{rk} \mathcal{J}_y F = \operatorname{res.rk} \mathcal{J}_y F = r$ den $y \in U$ guztietarako, U x-ren ingurune ireki bat izanik. Orduan, F azpimurgilketa da.*

Froga. Izan bitez $n = \dim_x N$ eta $m = \dim_{F(x)} M$. Emaitza lokala denez, koordenatutan lan eginez, demagun $x = \mathbf{0} \in U = \left(\mathfrak{m}^L\right)^{(n)} \subseteq_o N$, $F(x) = \mathbf{0} \in V = \left(\mathfrak{m}^L\right)^{(m)} \subseteq_o M$ eta $F \in R[[X_1, \ldots, X_n]]^{(m)}$ direla. Halaber, sinpletasunagatik suposa dezagun $\tilde{F} = (F_1, \ldots, F_r)$ bada, $\operatorname{res.rk} \mathcal{J}\tilde{F} = r$ dela. Hori dela eta,

$$\{F_1, \ldots, F_r, \pi_{r+1}, \ldots, \pi_n\}$$

funtzio familia N-ren koordenatu sistema da U-n. Beraz, $U = \left(\mathfrak{m}^L\right)^{(r)} \times \left(\mathfrak{m}^L\right)^{(n-r)}$ hartu eta koordenatu aldaketa baten ondotik $\tilde{F}(x_1, x_2) = x_1$ dela suposa dezakegu, hau da,

$$F(x_1, x_2) = (x_1, \psi(x_1, x_2)),$$

da $\psi \in R[[X_1, \ldots, X_n]]^{(n-r)}$ funtzio egoki batentzat eta $\operatorname{res.rk} \mathcal{J}_{\mathbf{0}} \psi = 0$ da. Bukatzeko, 0-ren ingurune batean ψ berretura serie tupla azken $n-r$ aldagaiekiko independentea dela frogatu behar dugu.

Alde batetik, $\partial_2 \psi = \mathbf{0}$ da U-n, non $\partial_2 \psi$-ren bidez azken $n-r$ aldagaiekiko deribatu formalen matrizea adierazten dugun. Ez balitz hala, (1.10) dela eta, $y \in U$ existituko litzateke non $\operatorname{rk} \mathcal{J}_y F > r$ den, hasierako hipotesiarekin kontraesana dena. Hori dela eta, $\operatorname{char} R = 0$ denez, Lema 1.41en ondorioz, ψ azken $n-r$ aldagaiekiko independentea da, hau da,

$$F(x_1, x_2) = (x_1, \psi(x_1))$$

da. Beraz, izan bedi $\pi\colon \left(\mathfrak{m}^N\right)^{(n)} \to \left(\mathfrak{m}^N\right)^{(r)}$ lehenengo r aldagaietara proiekzioa. Orduan, $F = (\mathrm{Id} \times \psi) \circ \pi$ da, eta, hortaz, F lokalki azpiraketa baten eta murgilketa baten konposizioa da. $\square$

1.5 Barietateen eraikuntza

Atal honetan jatorrizko barietate R-analitiko batetik abiatuta barietate berriak nola eraiki aztertuko dugu, bai oinarriko koefiziente eraztuna aldatuz, bai azpibarietate eta zatidura barietate kontzeptuak garatuz.

1.5.1 Eskalare murrizketa

Azpiatal honetan koefiziente eraztunaren azpieraztun baten gainean barietate egitura nola eratorri erakutsiko dugu. Horretarako, [24, Adibidea 13.6]n ageri den prozedura jarraituko dugu.

Izan bitez $(R, \mathfrak{m})$ pro-p domeinua eta Q azpieraztuna. Demagun Q bere kabuz pro-p domeinua dela $\mathfrak{n} := \mathfrak{m} \cap Q$ ideal maximalarekin. Demagun areago R finituki sortutako Q-modulu askea dela. Adibidez, Cohenen Egitura Teorema dela eta, $\dim_{\mathrm{Krull}}(R) = 1$ bada, R finituki sortutako $\mathbb{Z}_p$-modulua da $\operatorname{char} R = 0$ denean, edo finituki sortutatako $\mathbb{F}_p[[t]]$-modulua da $\operatorname{char} R = p$ positiboa denean.

Izan bitez M barietate R-analitikoa eta $\sigma\colon R \to Q^{(e)}$ Q-modulu isomorfismoa, hau da, finkatu $\{v_1, \ldots, v_e\}$ oinarria R-rentzat Q-modulu gisa eta definitu σ honela:

$$\sigma\left(\sum_{i=1}^{e} q_i v_i\right) = (q_1, \ldots, q_e).$$

Horrela, (U, ϕ, n) M-ren R-karta bakoitzerako, M-ren $\left(U, \sigma^{(n)} \circ \phi, ne\right)$ Q-karta dugu. Azken batean, $\left(\sigma^{(n)} \circ \phi\right)(U) \subseteq Q^{(ne)}$ azpimultzo irekia dela bakarrik frogatu behar da. Lehenik eta behin, deitu $\tau = \sigma^{-1}$, orduan, $\tau\left(\left(\mathfrak{n}^N\right)^{(e)}\right) \subseteq \mathfrak{m}^N$ da $N \in \mathbb{N}$ guztietarako. Bigarrenik, hartu $x \in \phi(U)$ elementua, $\phi(U)$ irekia denez $R^{(n)}$-n, existitzen da $N \in \mathbb{N}$ non

$$x + \left(\mathfrak{m}^N\right)^{(n)} \subseteq \phi(U)$$

den, eta, beraz,

$$\sigma^{(n)}(x) + \left(\mathfrak{n}^N\right)^{(ne)} \subseteq \sigma^{(n)}\left(x + \left(\mathfrak{m}^N\right)^{(n)}\right) \subseteq \left(\sigma^{(n)} \circ \phi\right)(U)$$

da.

Halaber, modu horretan eraikitako $\big(U, \sigma^{(n)} \circ \phi, ne\big)$ eta $\big(V, \sigma^{(n)} \circ \psi, ne\big)$ Q-kartak bateragarriak dira (ohartu bateragarritasuna $U \cap V \neq \varnothing$ denean soilik aztertu behar dela, eta, hortaz, Korolarioa 1.30en arabera, kartek dimentsio bera izan behar dute). Horrela,

$$\left(\sigma^{(n)} \circ \phi\right) \circ \left(\sigma^{(n)} \circ \psi\right)^{-1} = \sigma^{(n)} \circ (\phi \circ \psi^{-1}) \circ \left(\sigma^{-1}\right)^{(n)}$$

da eta $\phi \circ \psi^{-1}$ funtzioa R-analitikoa da, horrenbestez, bateragarritasuna hurrengo lemaren ondorioa da:

Lema 1.43 (cf. [24, Ariketa 13.4])**.** *Izan bedi $F \in \Lambda_0(R)[[X_1, \ldots, X_n]]$, orduan*

$$\sigma \circ F \circ \left(\sigma^{-1}\right)^{(n)} : \mathfrak{n}^{(en)} \to Q^{(e)}$$

funtzioa hertsiki Q-analitikoa da.

Froga. Ohiko moduan izan bedi $\Lambda(R) = \mathrm{Frac}(R)$ R INDa bada eta $\Lambda(R) = R$ bestalde, eta berdina $\Lambda(Q)$-rako.

Ohartu R INDa denean, R eraztuna Q-ren hedadura integrala denez,

$$\dim_{\mathrm{Krull}}(Q) = \dim_{\mathrm{Krull}}(R) = 1$$

dela, eta, beraz, Q pro-p domeinua INDa da. Ondorioz, π eta ρ hurrenez hurren R eta Q-ren uniformizatzaileak badira, orduan $\rho = \pi^N$ da $N \in \mathbb{N}$ batentzat.

Izan bedi $\{v_1, \ldots, v_e\}$ multzoa σ isomorfismoari dagokion R-ren oinarria Q-modulu gisa eta izendatu $\tau = \sigma^{-1}$. Halaber, suposa dezagun

$$F(X_1, \ldots, X_n) = \sum_{\alpha \in \mathbb{N}_0^{(n)}} a_\alpha X_1^{\alpha_1} \ldots X_n^{\alpha_n} \in \Lambda_0(R)[[X_1, \ldots, X_n]]$$

dela. Ikusi behar dugu existitzen direla $F_l^* \in \Lambda_0(Q)[[X_1, \ldots, X_{en}]]$, $l \in \{1, \ldots, e\}$, berretura serie formalak non

$$F \circ \tau^{(n)}(y_1, \ldots, y_{en}) = \sum_{l=1}^{e} v_l F_l^*\left(y_1, \ldots, y_{en}\right) \;\; \forall y_j \in \mathfrak{n}$$

den. Hots, $x_j = \sum_{i=1}^{e} v_i y_{ij}$ bada $y_{ij} \in \mathfrak{n}$ eta $j \in \{1, \ldots, n\}$ batzuentzat, orduan

$$F(x_1, \ldots, x_n) = \sum_{l=1}^{e} v_l F^*(y_{11}, \ldots, y_{en})$$

da. Lehenik eta behin, ohartu existitzen direla $a_\alpha(k,l) \in \Lambda(Q)$ non

$$a_\alpha v_k = \sum_{l=1}^{e} a_\alpha(k,l) v_l,$$

den $\alpha \in \mathbb{N}_0^{(d)}$ eta $k \in \{1,\ldots,e\}$ guztietarako. Aurrekoa R INDa denean frogatzeko, $F \in \Lambda_0(R)[[\mathbf{X}]]$ eta $\rho = \pi^N$ direla hartu behar da kontuan, eta, beraz, existitzen dela $L \in \mathbb{N}$ zenbaki osoa $a_\alpha \rho^{L|\alpha|} \in R$ izanik $\alpha \in \mathbb{N}_0^{(d)}$ guztietarako. Bereziki,

$$a_\alpha(k,l)\rho^{|\alpha|L} \in Q \ \forall k \in \{1,\ldots,e\}. \tag{1.11}$$

Horrela, Teorema multinomiala aplikatuz, edozein $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}_0^{(n)}$ tuplatarako existitzen dira $\gamma_k(\beta) \in Q$ elementuak non

$$\prod_{j=1}^{n}\left(\sum_{i=1}^{e} v_i Y_{ij}\right)^{\alpha_j} = \sum_{|\beta|=|\alpha|}\sum_{k=1}^{e}\gamma_k(\beta) v_k \prod_{i=1}^{e}\prod_{j=1}^{n} Y_{ij}^{\beta_{ij}}$$

den (Y_{ij}-ak aldagaiak dira). Azkenik, bila genbiltzan berretura serie formalak

$$F_l^*(Y_{11},\ldots,Y_{en}) = \sum_{\alpha\in\mathbb{N}_0^{(d)}}\sum_{|\beta|=|\alpha|}\sum_{k=1}^{e} a_\alpha(k,l)\gamma_k(\beta)\prod_{i=1}^{e}\prod_{j=1}^{n} Y_{ij}^{\beta_{ij}}$$

dira. Horrenbestez, R ez denean INDa, berehalakoa da aurreko berretura serieen koefizienteak Q-n daudela, eta, aldiz, R INDa denean, $F_l^* \in \Lambda_0(Q)[[\mathbf{X}]]$ da (1.11) dela eta. □

Gainera, aurreko barietate Q-analitiko egitura hautatutako σ isomorfismoarekiko independentea da, hau da, R-ren Q-modulu oinarriarekiko independentea. Azken batean, nahikoa da $\mathfrak{m}^N$ barietate R-analitikoan σ eta $\tilde{\sigma}$ Q-modulu isomorfismoek Q-karta baliokideak sortzen dituztela ikustea. Horretarako ohartu

$$\tilde{\sigma}\circ\sigma^{-1}\colon \sigma\left(\mathfrak{m}^N\right) \to \tilde{\sigma}\left(\mathfrak{m}^N\right)$$

ez dela aplikazio lineal bat besterik, $A \in \mathrm{GL}_e(Q)$ oinarri aldaketa matrizeak definiturikoa hain zuzen ere, eta, beraz, $\tilde{\sigma}\circ\sigma^{-1}$ funtzio Q-bianalitikoa da. Hortaz, barietate berriak sortzeko prozedura honi besterik gabe *eskalare murrizketa* deituko diogu, hau da, isomorfismoa zehaztu gabe. Bukatzeko, jar dezagun arreta kasu partikular batean:

Korolarioa 1.44. *Izan bedi R bat Krull dimentsioko pro-p domeinua. Orduan, talde R-analitiko oro eskalareak murriztuz talde p-adiko analitikoa da* $\mathrm{char}(R) = 0$ *bada, edo talde* $\mathbb{F}_p[[t]]$*-analitikoa da* $\mathrm{char}(R) = p$ *positiboa bada.*

Zenbait baldintza gehigarrekin aurrekoaren alderantzikoa ere egia da.

Teorema 1.45. *Izan bedi G talde R-analitiko ez-diskretua.*

(i) (cf. [24, Teorema 13.23]) *Demagun G-k talde p-adiko analitiko egitura onartzen duela, orduan R finituki sortutako* $\mathbb{Z}_p$*-modulua da.*

(ii) (cf. [45, Teorema 1.1]) *Demagun G finituki sortua (talde topologiko gisa) dela eta G-k talde* $\mathbb{F}_p[[t]]$*-analitiko egitura onartzen duela, orduan, R finituki sortutako* $\mathbb{F}_p[[t]]$*-modulua da.*

Bigarren erreferentzian, autoreek beraiek erakusten dute finituki sortua izatearen baldintza beharrezkoa dela, izan ere $\mathbb{F}_p[[t_1]]$ eta $\mathbb{F}_p[[t_1, t_2]]$ talde topologiko batukorrak elkarri isomorfoak dira. Aitzitik, finituki sorturiko taldeetan galdera naturala da egitura analitikoak oinarriko eraztuna zenbateraino zehazten duen. Horrela tentagarria da honakoa espekulatzea:

Aierua 1.46. Demagun G finituki sortutako talde topologiko ez-diskretuak talde R-analitiko eta talde Q-analitiko egiturak onartzen dituela. Orduan, R eta Q eraztunek Krullen dimentsio eta karakteristika bera dute.

1.5.2 Azpibarietateak

Pro-p domeinu orokorren gainean azpibariete kontzeptzua lantzeko dago. Gorputz lokalen gaineko barietate analitikoen teorian hainbat definizio baliokide daude (ikusi [68, 89. orria]). Hemen horietako batzuk landuko ditugu, betiere gogoan izanda zein-nahi barietate definiziok aintzat eduki behar duela azpimultzoa bere kabuz barietate R-analitikoa dela.

Definizioa 1.47. Izan bedi $F \colon N \to M$ murgilketa ahul injektiboa. Orduan, $F(N)$ M-ren *azpimultzo murgildua* da.

Adibidez, izan bedi $\mathfrak{m}$ $\mathbb{Z}_p[[t]]$-ren ideal maximala barietate $\mathbb{Z}_p[[t]]$-analitiko egitura naturalarekin, eta hornitu $t\mathfrak{m}$ multzoa egitura $\mathbb{Z}_p[[t]]$-analitiko batez $\psi \colon t\mathfrak{m} \to \mathfrak{m}$, $tx \mapsto x$ karta globalarekin. Orduan, $\iota \colon t\mathfrak{m} \to \mathfrak{m}$ partekotasuna murgilketa ahula da, $\mathcal{J}_x(\mathrm{Id} \circ \iota \circ \psi^{-1}) = (t)$ baita $x \in t\mathfrak{m}$ guztietarako. Aitzitik, adibide honetan, $t\mathfrak{m}$-ren topologia, apropos ψ homeomorfismo egiteko aukeratua, ez dator azpiespazio topologiarekin bat. Egitura topologikoen arteko bateragarritasuna edukitzea komeni denez, honakoa defini daiteke:

Definizioa 1.48. Izan bedi M barietate R-analitikoa. Orduan, $S \subseteq M$ *azpibarietate R-analitikoa* da $s \in S$ guztietarako existitzen badira $k_s \in \mathbb{N}_0$, s-ren U_s ingurune irekia eta s-ren (V_s, ϕ_s, d_s) R-karta M-n non

- $U_s \subseteq V_s$ eta
- $\phi_s(U_s) = \phi_s(V_s) \cap \left(R^{(k_s)} \times \{0\}^{(d_s - k_s)}\right)$

diren. *Azpibarietatearen dimentsioa* s-n, $\dim_s S$ adieraziko dena, k_s zenbakia da.

Azpiespazio topologiarekin S barietate R-analitikoa da, $s \in S$ puntu bakoitzari $(U_s, (\phi_{s,1}, \ldots, \phi_{s,k_s}), k_s)$ karta eslei baitiezaiokegu. Lema 1.38 aplikatuz honakoa dugu:

Proposizioa 1.49. *Izan bedi $F\colon N \to M$ funtzio R-analitiko injektiboa. Orduan, $F(N)$ M-ren azpibarietate R-analitikoa da baldin eta soilik baldin F murgilketa bada.*

Oharra 1.50. Ohartu azpibarietate R-analitikoak lokalki itxiak direla. Izan ere, M barietate R-analitikoa eta $S \subseteq M$ azpibarietatea badira, $s \in S$ bakoitzerako existitzen dira $U_s \subseteq_o S$ eta $V_s \subseteq_o M$ azpimultzoak non $\phi_s(U_s)$ ekuazio lineal batzuen bitartez definituta dagoen $\phi_s(V_s)$-n. Bereziki, U_s itxia da V_s-n, hau da, $V_s \setminus U_s$ irekia da V_s-n. Hortaz, $V = \cup_{s\in S} V_s$ eta $S = \bigcup_{s\in S} U_s$ direnez, $V \setminus S = \bigcup_{s\in S} V_s \setminus U_s$ irekia da V-n, hau da, S itxia da V multzo irekian.

Hau ere berehalakoa da:

Lema 1.51. *Izan bitez M barietate R-analitikoa eta S azpibarietatea. Demagun* $\dim_s M = \dim_s S$ *dela $s \in S$ guztietarako. Orduan, S irekia da M-n.*

Froga. Definizioa 1.48ko notazioa mantenduko da. Horrela, $k = d_s$ eta ϕ homeomorfismoa direnez, $U_s = V_s$ da. Beraz, $S = \bigcup_{s\in S} V_s$ irekia da M-n. □

Definizioa 1.48 [27]n emandako azpibarietatearen definizio hau baino gogorragoa da:

Definizioa 1.52. Izan bedi M barietate R-analitikoa. Orduan, $S \subseteq M$ *azpibarietate R-analitiko ahula* da $s \in S$ bakoitzerako existitzen direnean s-ren U_s ingurune irekia S-n, s-ren (V_s, ϕ_s, d_s) R-karta M-n eta $E_s \leq K^{(d_s)}$ espazio K-bektoriala, $K = \operatorname{Frac} R$ izanik, non

- $U_s \subseteq V_s$ eta

- $\phi_s(U_s) = \phi_s(V_s) \cap E_s$

diren.

Gorputzen gainean Definizioa 1.48 eta Definizioa 1.52 baliokideak dira. Izan ere, oinarri aldaketa egoki baten ondotik $E_s = K^{(k)} \times \{\mathbf{0}\}$ dela suposa daiteke. Haatik, etsenplu batekin erakutsiko denez, orokorrean ez dira baliokideak. Izan bitez $R = \mathbb{Z}_2[[t]]$ pro-2 domeinua, $\mathfrak{m} = (2, t)R$ ideal maximala, $K = \mathbb{Q}_2((t))$, $M = \mathfrak{m}^{(2)}$ barietate R-analitikoa eta

$$E = \left\{(x, y) \in K^{(2)} \mid 2x - ty = 0\right\}$$

espazio K-bektoriala. Orduan, $M \cap E = \{(ta, 2a) \mid a \in R\}$ multzoa argi eta garbi azpibarietate R-analitiko ahula da, eta

$$\psi\colon M \cap E \to R, (ta, 2a) \mapsto a$$

R-karta globala definitu daiteke. Aitzitik, $\mathcal{J}_s\left(\mathrm{Id} \circ \iota \circ \psi^{-1}\right) = \begin{pmatrix} t \\ 2 \end{pmatrix}$ da $s \in M \cap E$ guztietarako, eta, beraz, ι murgilketa ahula da, baina ez murgilketa. Azken batean, $K^{(2)}$ espazioan oinarri kanonikotik $\beta = \{(t, 2), (0, 1)\}$-ra oinarri aldaketa funtzioa, deitu $L_A\colon K^{(2)} \to K^{(2)}$,

$$A = \begin{pmatrix} 1/t & 0 \\ -2/t & 1 \end{pmatrix}$$

oinarri aldaketa matrizeak definitzen du eta funtzio horrekin $L_A(E) = K^{(1)} \times \{0\}$ da. Hala eta guztiz ere, L_A ez da funtzioa R-bianalitikoa. Bide batez, horrek erakusten du batzuetan azpibarietate ahulentzat koordenatu sistema zehaztea nekeza izan daitekela. Gorputzen gainean hori egiteko era naturala E_s-ren $\beta = \{v_1, \ldots, v_k\}$ oinarria finkatu eta "karta" gisa

$$\psi\colon \phi_s(V_s) \cap E_s \to K^{(k)},\ \sum_{i=1}^{k} \alpha_i v_i \mapsto (\alpha_1, \ldots, \alpha_k) \in K^{(k)}$$

funtzioa erabiltzea da. Baina, oinarri batzuetatako gerta daiteke $\operatorname{im}\psi \not\subseteq R^{(k)}$ izatea, eta, beraz, gerta daiteke ψ R-karta ez izatea.

Gorputzen gaineko Lieren teoria klasikoan, azpibarietate murgilduak (Definizioa 1.47ren baliokidea) eta azpibarietate enbebituak (Definizioa 1.48ren baliokidea) bereizten dira (ikusi [51, Kapitulua 5]). Bigarrena oro har definizio gogorragoa da, baina barietatea trinkoa denean bi kontzeptuak bat datoz, baita INDak diren pro-p domeinuen gaineko barietate analitikoetan ere (ikusi [68, Zatia II, Atala III.11.2]). Pro-p domeinu orokorren gainean, alabaina, ikusi da darabilgun azpibaritate kontzeptua garbi zehaztu behar dela.

Bestetik, [45, Atala 4]n, azpibarietatate $\mathbb{F}_q[[t]]$-analitikoak funtzio analitikoen zuntz gisa karakterizatzen dira. Konkretuago, honakoa frogatzen da:

Proposizioa 1.53 (ikusi [45, Korolarioa 4.2])**.** *Izan bitez M barietate $\mathbb{F}_q[[t]]$-analitikoa eta $S \subseteq M$ azpimultzoa. Demagun*

(i) *S homogeneoa dela, hau da, S orbita bakarraren parte da M-ren automorfismo R-bianalitikoen ekintzarekiko, eta*

(ii) *S azpimultzo analitikoa dela, hau da, $s \in S$ bakoitzerako existitzen dira U ingurune irekia eta $\{f_i\colon U \to \mathbb{F}_q[[t]]\}_{i\in I}$ funtzio $\mathbb{F}_q[[t]]$-analitikoak*

$$S \cap U = \{x \in U \mid f_i(x) = 0 \ \forall i \in I\}$$

izanik.

Orduan, S azpibarietate $\mathbb{F}_q[[t]]$-analitikoa da.†

Ohartu M talde R-analitikoa denean, ezker biderketa funtzioen ekintza iragankorra dela, eta, ondorioz, kasu horretan azpimultzo guztiak homogeneoak dira.

Domeinu orokorretan emaitza honen parekoa frogatzeko, M barietate R-analitikoa eta $S \subseteq M$ badira, S *azpimultzo R-analitikoa* dela diogu $s \in S$ guztietarako existitzen badira U ingurune irekia eta $\{f_i\colon U \to R \mid i = 1, \dots, r_s\}$ funtzio R-analitikoak

$$S \cap U = \{y \in U \mid f_i(y) = 0 \ \forall i = 1, \dots, r_s\}$$

izanik. Alegia, azpimultzo analitiko bat lokalki funtzio analitiko batzuen zeroen multzoa da. Alde batetik, kontzeptu honek azpibarietate nozioa orokortzen du; eta bestetik, $R[[X_1, \dots, X_n]]$ eraztun noetherdarra denez (ikusi [49, Teorema IV.9.4]), definizioa erlaxatu daiteke r_s infinitua izatea baimenduz.

†Oinarriko eraztuna IND denez, emaitza honek azpibarietateen edozein definiziotarako balio du.

Definizioa 1.54. Izan bitez M barietate R-analitikoa eta $S \subseteq M$ azpimultzo R-analitikoa. Orduan, $s \in S$ puntua *erregularra* dela diogu existitzen badira s-ren U ingurune irekia eta $\{f_i \colon U \to R \mid i = 1, \ldots, r_s\}$ funtzio R-analitikoak honakoa betez:

(i) $S \cap U = \{y \in U \mid f_i(y) = 0 \ \forall i = 1, \ldots, r_s\}$ da eta

(ii) $F := (f_1, \ldots, f_{r_s})$ bada, orduan $\operatorname{res.rk}(\mathcal{J}_s F) = r_s$ da.

Definizioko r_s zenbaki osoari S-ren *koheina* deritzo, eta Lema 1.38 dela eta, $r_s \leq \dim_s S$ da.

Lema 1.55. *Izan bitez M barietate R-analitikoa eta $S \subseteq M$. Orduan, S azpibarietatea da baldin eta soilik baldin S-ko puntu guztiak erregularrak badira.*

Froga. Alde batetik, *soilik baldina* berehalakoa da definizioa eta Lema 1.38 kontuan hartuta. Bestetik, *baldina* frogatzeko, izan bitez $x \in S$ eta $d = \dim_x S$. Suposa dezagun x puntua erregularra dela eta r koheina duela, orduan existitzen da x-ren U ingurunea non

$$S \cap U = \{y \in U \mid f_i(y) = 0 \ \forall i = 1, \ldots, r\}$$

den. Izan bedi $F = (f_1, \ldots, f_r)$. Horrela, $\operatorname{res.rk} \mathcal{J}_x F = r$ denez, $\{f_i\}_{i=1}^r$ familia M-ren $\{f_i\}_{i=1}^d$ koordenatu sistema batera hedatu daiteke. Koordenatu sistema horrekin, $U' \subseteq_o U$ azpimultzo ireki txikiago baterako

$$S \cap U' = \{y \in U' \mid f_1(y) = \cdots = f_r(y) = 0\}$$

da, hau da, M azpibarietatea da eta $d - r$ dimentsioa du x-n. □

Azkenik, talde R-analitikoetan talde eta barietate azpiegiturak uztar ditzakegu. Hots,

Definizioa 1.56. Izan bedi G talde R-analitikoa. *Azpitalde analitiko* bat azpibarietate R-analitikoa ere baden $H \leq G$ azpitaldea da.

Lema 1.57. *Azpitalde R-analitikoak itxiak dira.*

Froga. Jakina da talde topologiko batean azpitalde lokalki itxiak benetan itxiak direla, eta, beraz, emaitza Oharra 1.50en ondorioa da. Konkretuago, H lokalki itxia denez, H irekia da $\overline{H}$ itxitura topologikoan. Demagun, absurdura eramanez, $H \neq \overline{H}$ dela, bereziki existitzen dela $gH \subseteq \overline{H}$ ezker koklase ez-tribiala. Horrela, H eta gH azpimultzo irekiak $\bar{H}$-n dentsoak direnez, $gH \cap H \neq \varnothing$ da, koklaseak disjuntuak izatearekin kontraesana dena. □

1.5.3 Zatidura analitikoak

Ideal nagusietako domeinuen gainean zatidura analitikoei buruzko informazioa [68, Zatia II, Atala III.12]n topa daiteke. Halaber, [11, Kapitulua III, §1.6]n talde $\mathbb{F}_p[[t]]$-analitiko bat azpitalde analitiko batekin zatitzean zatidurak egitura $\mathbb{F}_p[[t]]$-analitikoa duela frogatzen da. Domeinu orokorretan emaitza hori ohiko argudioekin froga daiteke.

Lema 1.58. *Izan bitez G d dimentsioko talde R-analitikoa, $N \trianglelefteq G$ k dimentsioko azpitalde R-analitiko normala eta* $\mathrm{pr}\colon \mathfrak{m}^{(d)} \to \mathfrak{m}^{(d-k)}$ *proiekzioa azken $d-k$ koordenatuetara. Orduan, existitzen da identitatearen (V, ψ) R-karta non* $\mathrm{pr} \circ \psi(x) = \mathrm{pr} \circ \psi(y)$ *den baldin eta soilik baldin $xy^{-1} \in N$ bada.*

Notazioa. Aurreko lemakoa bezalako (V, ψ) R-karta N-ra *moldatua* dagoela esaten da.

Froga. Lehenik eta behin, finka dezagun notazioa: $r = d - k$, $\mathrm{pr}_1\colon \mathfrak{m}^{(d)} \to \mathfrak{m}^{(k)}$ eta $\mathrm{pr}_2\colon \mathfrak{m}^{(d)} \to \mathfrak{m}^{(r)}$, hurrenez hurren, lehenbiziko k eta azken r koordenatuetara proiekzioak dira, hau da, enuntziatuko notazioarekin $\mathrm{pr}_2 = \mathrm{pr}$ da. Horrela, N azpibarietatea denez, existitzen da identitatearen (U, ϕ) R-karta G-n non

$$N \cap U = \{x \in U \mid \phi_{k+1}(x) = \cdots = \phi_d(x) = 0\}$$

den. Gainera, Lema 1.21 dela eta, (U, ϕ) d dimentsioko talde R-estandarra dela suposa dezakegu, eta ϕ erabilita $\left(\mathfrak{m}^M\right)^{(d)}$ multzoarekin identifikatu daiteke, $M \in \mathbb{N}$ baterako. Horrela, talde biderketa $\mathbf{F}$ talde eragiketa formalak ematen du eta identitatea $\mathbf{0}$ da. Hortaz,

$$N \cap U = \{y \in U \mid y_{k+1} = \cdots = y_d = 0\}$$

da, eta $(N \cap U, \phi_1)$, non $\phi_1 = \mathrm{pr}_1|_U$, $\mathbf{0}$-ren R-karta da N-n. Betalde, definitu barietate r-dimentsional hau:

$$W = \{y \in U \mid y_1 = \cdots = y_k = 0\},$$

(W, ϕ_2) R-kartarekin, $\phi_2 = \mathrm{pr}_2|_U$ izanik, eta hartu funtzio R-analitiko hau:

$$\begin{aligned} F\colon\ (N \cap U) \times W &\longrightarrow U \\ (n, w) &\longmapsto n \cdot w = \mathbf{F}(n, w). \end{aligned}$$

Ohartu $\mathcal{J}_{\mathbf{0}}\left(F \circ (\phi_1, \phi_2)^{-1}\right) = I_d$ dela, beraz Alderantzizko Funtzioaren Teoremaren arabera, F is lokalki funtzio R-bianalitikoa da, hau da, existitzen dira U_1

$\mathbf{0}$-ren ingurune irekia N-n, U_2 $\mathbf{0}$-ren ingurune irekia W-n eta U_3 $\mathbf{0}$-ren ingurune irekia U-n, non $F\colon U_1 \times U_2 \to U_3$ funtzio R-bianalitikoa den. Gainera, ingurune txikiagoak hartuz, $U_1 = \left(\mathfrak{m}^L\right)^{(d)} \cap N$ dela suposa dezakegu $L \geq M$ baterako, eta, bereziki, $U_1 \leq U$ dela.

Hartu orain $V = \left(\mathfrak{m}^K\right)^{(d)} \subseteq U_3$, $K \in \mathbb{N}$ egoki baterako, eta $\psi := (\phi_1, \phi_2)\circ F^{-1}|_V$. Ikus dezagun $\mathbf{0}$-ren (V, ψ) R-karta N-ra moldatua dagoela. Demagun, lehenengo, $x, y \in V$ direla non $xy^{-1} = n \in N$ den. Orduan, $x, y \in V \subseteq U_3$ direnez, existitzen dira $n_1, n_2 \in U_1$ eta $h_1, h_2 \in U_2$ elementu bakarrak non $x = n_1h_1$ eta $y = n_2h_2$ diren, eta, beraz, $n_1h_1 = x = nn_2h_2$ da. Horrela, $n = xy^{-1} \in V \cap N$ denez, $n \in U_1$ da, hau da, $nn_2 \in U_1$ da. Beraz, $F|_{U_1\times U_2}$ bijekzioa direnez, $h_2 = h_1$ dugu. Alderantziz, $\mathrm{pr}_2(x) = \mathrm{pr}_2(y)$ bada, orduan $x = n_1h$ eta $y = n_2h$ dira $n_1, n_2 \in U_1$ eta $h \in U_2$ batzuentzat, alegia $xy^{-1} = n_1n_2^{-1} \in N$. □

Proposizioa 1.59. *Izan bitez G talde R-analitikoa eta $N \trianglelefteq S$ azpitalde analitiko normala. Orduan, G/N zatidura taldean egitura analitiko bat existitzen da, eta egitura hori bakarra da zeinarekiko $\pi\colon G \to G/N$ zatidura epimorfismoa azpiraketa den.*

Froga. Finka dezagun notazioa: $d = \dim G$, $k = \dim N$ eta $\mathrm{pr}\colon \mathfrak{m}^{(d)} \to \mathfrak{m}^{(d-k)}$ azken $d-k$ koordenatuetara proiekzioa dira.

Lehenengo, G/N taldean atlas R-analitiko bat eman behar dugu. Horretarako nahikoa da $1N$-ren karta egoki bat ematea. Lema 1.58 dela eta, existitzen da N-ra moldatutako identitatean (V, ψ) R-karta G-n. Beraz, π funtzio irekia denez, $\overline{V} = \pi(V)$ irekia da G/N-n, eta karta N-ra moldatuta dagoenez, $\mathrm{pr}\circ\psi$ ongi definitua dago $\overline{V}$-n. Horrenbestez, $\left(\overline{V}, \mathrm{pr}\circ\psi\right)$ R-kartak $d-k$ dimentsioa du eta $1N \in \overline{V}$ da.

Behin R-karta hori izanda, $L_{aN}\colon G/N \to G/N$ ezker biderketa funtzioen bidez, $\left\{\left(L_{aN}(\overline{V}), \mathrm{pr}\circ\psi \circ L_{a^{-1}N}\right)\right\}_{a\in G}$ atlas R-analitikoa defini dezakegu. Karta horiek bateragarriak dira. Izan ere, jatorrizko karta N-ra moldatua zegoenez, $L_{aN}\left(\overline{V}\right) \cap L_{bN}\left(\overline{V}\right) \neq \varnothing$ denean, $K \in \mathbb{N}$ behar bezain handi baterako honakoa betetzen da:

$$(\mathrm{pr}\circ\psi \circ L_{a^{-1}N}) \circ (\mathrm{pr}\circ\psi \circ L_{b^{-1}N})^{-1}(x) = \mathrm{pr}\circ(\psi \circ L_{a^{-1}b} \circ \psi^{-1})(\mathbf{0}, x),\ \forall x \in \left(\mathfrak{m}^K\right)^{(d-k)} \quad (1.12)$$

–$L_{a^{-1}b}\colon G \to G$ ohiko ezker biderketa funtzioa da– eta, beraz, koordenatu aldaketa funtzioak ere funtzio R-analitikoak dira.

Halaber, $\pi\colon G \to G/N$ epimorfismoa identitatearen inguruan koordenatutan, (V, ψ) eta $\left(\overline{V}, \mathrm{pr}\circ\psi\right)$ kartekiko, pr baino ez da. Horrenbestez, π azpiraketa da identitatean, eta G talde R-analitikoa denez, π azpiraketa da.

Azkenik, demagun zatidura taldean bi egitura R-analitiko daudela $-(G/N)_1$ eta $(G/N)_2$ moduan adieraziko ditugu– eta dagozkien zatidura epimorfismoak azpiraketak direla. Lema 1.38ren arabera, existitzen dira R-kartak non $\pi\colon G \to (G/N)_i$ ($i \in \{1,2\}$) funtzioak koordenatutan $(x_1,\ldots,x_d) \to (x_{k+1},\ldots,x_d)$ diren. Horrela, R-karta horiekiko $\mathrm{Id}\colon (G/N)_1 \to (G/N)_2$ identitatea da koordenatutan. Alegia, Id funtzio R-bianalitikoa da eta barietate egitura bakarra da. □

Azkenik, erraz ikusten da aitzineko barietate egiturarekin zatidura taldean biderketa eta alderantzizkoa funtzio R-analitikoak direla ((1.12)ko modu berean frogatzen da). Hori dela eta, emaitza hau dugu:

Teorema 1.60. *Izan bitez G talde R-analitikoa eta $N \trianglelefteq G$ azpitalde analitiko normala. Orduan, G/N zatidura taldean egitura R-analitiko bat existitzen da, eta egitura hori bakarra da zeinarekiko $\pi\colon G \to G/N$ epimorfismo kanonikoa azpiraketa den.*

1.6 Oharrak

Talde analitikoak pro-p domeinu orokorren gainean lehenbiziko aldiz [24, Kapitulua 13]n definitu ziren, nahiz eta definizio horiek ideal nagusietako domeinu osoen gainean jada [11] eta [68]n agertu. Atala 1.1etik 1.3ra bitartean talde horien propietate orokorrak ageri dira, eta aurkezpenak batez ere [24, Kapitulua 13] darrio. Ekarpen original nagusia Azpiatala 1.3.1 da, bertan talde R-analitikoen talde egitura azpitalde R-estandar egoki baten bitartez deskribatzen da.

Talde eragiketa formal batetik abiatuz Lie aljebra bat eraikitzea [11]n dago eginda, eta Atala 1.4ko emaitza gehienek [68]n daudenak orokortzen dituzte, ideal nagusietako domeinuen ordez pro-p domeinu orokorrak hartuta.

Atala 1.5en egiten den azpibarietate eta zatidura barietate kontzeptuen garapena berria da, eta [24, 349 orria]n iradokita zegoen. Autorearen iritziz, kontzeptu horiek etorkizuneko ikerketetarako baliagarriak izan daitezke.

Atal honetan dagoeneko emandako erreferentziez gain, talde R-analitikoen propietate anitz topa daitezke [45] eta [54] artikuluetan.

Azkenik, eta albo iruzkin gisa, aipatu behar da (1.1)en, Λ-ren definizioak (IND-en eta INDak ez diren domeinuen arteko bereizketak) hasiera batean arbitrario samarra dirudiela. Baina, esate baterako, talde p-adiko analitikoen teorian, $\mathbb{Z}_p$ taldearen $L_p\colon \mathbb{Z}_p \to p\mathbb{Z}_p$ eta $\mathrm{Id}\colon \mathbb{Z}_p \to \mathbb{Z}_p$ R-kartak bateragarriak dira, eta barietate R-analitikoen teoria orokorrean hala izaten jarraitzea nahi dugunez, aurreko bereizketa beharrezkoa da. Azken batean, pro-p domeinuen gaineko teoria analitikoak, garatu ahala, orain arteko gorputz lokalen gaineko, $\mathbb{Q}_p$ kasu, teoria analitikoa *zabaldu* egin behar du. Halere, *R faktorizazio bakarreko domeinua* (FBD) denean, bereizketa hori ekidin daiteke beti $\Lambda = \mathrm{Frac}(R)$ hartuaz. Hori eginez gero, $K[[\mathbf{X}]]$-ko berretura serie konbergenteen multzoa $R[[\mathbf{X}]]$ da INDak ez diren faktorizazio bakarreko pro-p domeinuetarako. Aitzitik, orokorrean egoera ez dago argi. Dena den, komeni da [24, Atala 13.6] irakurtzea, honen inguruko eztabaida sakonagoa baitago bertan.

2

Linealtasuna talde R-analitiko trinkoetan

Izan bedi G taldea, K gorputzaren gaineko n mailako *adierazpen leial* bat $\rho\colon G \to \mathrm{GL}_n(K)$ talde monomomorfismo bat da, edo, baliokideki, n dimensiontsoko W K-espazio bektorial baten gaineko G-ren ekintza leial bat.

Definizioa 2.1. Talde bat *lineala* da haren adierazpen lineal leial bat exisitzen denean.

Horrenbestez, talde linealak matrize alderanzgarrien taldeen azpitaldeak direla pentsa daiteke. Horrela, talde horiek matrize taldeetarako ezagunak diren hainbat propietate dituzte, eta horiekin lan egiterakoan aljebra eta geometria uztartzen dira. Talde linealak Jordanek [46] aurkeztu zituen, eta ondorengo lanetan Jordan berak eta beste matematikari nabarmen anitzek beren azterteka sistematikoa sustatu zuten. Geroztik, talde linealen teoria biziki garatu da, eta aplikazio andana ditu talde teorian bertan zein bertatik kanpo.

Oinarrizko ohar pare bat egin behar ditugu:

Oharra 2.2. (i) Adirazpen linealak gorputzen gainean definitu beharrean, integritate domeinuen gainean ere defini daitezke. Haatik, ohartu G-k integriate domeinu baten gainean adierazpen leial bat badu, gorputz baten gainean ere baduela halakorik.

(ii) Talde birtualki linealak linealak dira. Izan ere, izan bedi $H \leq G$ indize finituko azpitaldea eta demagun H-k W K-bektore espazioan ekiten duela. Hartu H-ren ezker transbertsal bat G-n, deitu T, eta definitu

$$V := \bigoplus_{t \in T} W_t,$$

non $W_t \cong W$ den. Bereziki, V bektore espazioa da eta $|G : H| \dim W$ dimentsioa du. Notazioa erraztearren izendatu (t, w) moduan $w \in W$-ri W_t-n dagokion elementua. Orduan, G-k V-ren gainean ekiten du:

$$g \cdot (t, w) = \left(\widetilde{gt}, w \cdot h\right),$$

hemen $\widetilde{gt}$ elementua T-n gtH koklasearen ordezkaria da, hau da, $\widetilde{gt} \in T$ eta $h = gt\widetilde{gt}^{-1} \in H$ betetzen duen elementu bakarra da. Erraz egiazta daiteke horrek G-ren adierazpen leial bat definitzen duela, *adierazpen eratorria* deituko duguna.

Talde R-analitikoei dagokionez, 1965ean Lazardek talde p-adiko trinkoak linealak direla frogatu zuen. Talde R-analitikoak talde p-adikoen orokorpena direnez, naturala da bigarrenetan berdina gertatzen ote den galdetzea. Alde batetik, Lieren aljebretan eskuragarri dagoen adierazpen adjuntuaren (ikusi Atala A.2) analogoa dugu, hau da, talde R-estandarrak "modulo zentroa" linealak dira.

Proposizioa 2.3 (cf. [42, Proposizioa 5.1])**.** *Izan bedi S talde R-estandarra, orduan $S/Z(S)$ lineala da R-ren gainean.*

Froga. Lehenik, S talde R-estandarra denez, $\left(\mathfrak{m}^N\right)^{(d)}$-rekin identifika dezakegu, d taldearen dimentsioa eta N maila izanik. Horrela, izan bitez $\mathbf{F}$ eta $\mathbf{I}$ hurrenez hurren S-ren talde eragiketa formala eta alderantzizko formala. Orduan, c_g konjokazio funtzioa hertsiki analitikoa da, eta

$$\mathbf{C}_g(\mathbf{X}) = (C_1^g(\mathbf{X}), \ldots, C_d^g(\mathbf{X})) := \mathbf{F}(\mathbf{F}(\mathbf{I}(g), \mathbf{X}), g)$$

berretura serie formalen tuplak emanda dago (hemen $\mathbf{X}$ aldagaien d-tupla da).

Eman dezagun S-ren ekintza bat finituki sortutako W R-modulu aske batean. Lehenengo, S-k $R[[\mathbf{X}]]$-n ekiten du $g \cdot f(\mathbf{X}) := f \circ \mathbf{C}_g(\mathbf{X}) \in R[[\mathbf{X}]]$ moduan. Bereziki, $\pi_i \in R[[\mathbf{X}]]$ berretura serie formala $\pi_i(\mathbf{X}) = X_i$, $i$garren koordenatura proiekzio funtzioa, bada; (1.5)en arabera, badaude $\pi_{i,\alpha} \in R[[\mathbf{X}]]$ non

$$(g \cdot \pi_i)(\mathbf{X}) = C_i^g(\mathbf{X}) = X_i + \sum_{|\alpha| \geq 1} \pi_{i,\alpha}(g) X_1^{\alpha_1} \ldots X_d^{\alpha_d}$$

den. Izan bedi $\mathcal{I}$

$$\left\{\pi_{i,\alpha} \mid \alpha \in \mathbb{N}_0^{(d)}, i \in \{1, \ldots, d\}\right\} \tag{2.1}$$

berretura serie formalek sorturiko $R[[\mathbf{X}]]$-ren ideala. Horrela, $R[[\mathbf{X}]]$ eraztun noetherdarra denez (ikusi [49, Teorema 9.4]), $\mathcal{I}$ ideala (2.1)eko azpimultzo finitu batek, deitu $\mathcal{F}$, sortzen du. Hartu $m = \max\{|\alpha| \mid \pi_{i,\alpha} \in \mathcal{F}\} \in \mathbb{N}$ eta izendatu $\mathfrak{M}$ moduan $(X_1, \ldots, X_d)R[[\mathbf{X}]]$ ideala, orduan $W = \mathfrak{M}/\mathfrak{M}^{m+1}$ heina finituko R-modulu askea da eta naturalki S-k W-ren gainean eragiten du.

Ikus dezagun ekintza horren nukleoa $Z(S)$ dela, hau da, $S/Z(S)$-ren ekintza leiala dela W-n. Alde batetik, agerikoa da $Z(S)$-ren ekintza tribiala dela W-n. Kontrako norantzan, demagun g elementuaren ekintza tribiala dela W-n. Alegia, $\pi(g) = 0$ da $\pi \in \mathcal{F}$ guztietarako, eta, beraz, $\pi_{i,\alpha}(g) = 0$ da $i \in \{1, \ldots, d\}$ eta $\alpha \in \mathbb{N}_0^{(d)}$ guztietarako. Hortaz, $g \cdot \pi_i(\mathbf{X}) = X_i$ da, hau da, $g \in Z(S)$ da. □

Beste alde batetik, Lazarden froga –Atala 2.1en laburki azalduko dugu– Adoren Teoreman eta Baker-Campbell-Hausdorffen formulan oinarritzen da. Ideia horretan berean oinarrituta, Camina eta Du Sautoyk [14] talde $\mathbb{Z}_p[[t]]$-estandar perfektuak linealak direla frogatu zuten –N mailako S talde R-estandarra perfektua da $S' = S_{2N}$ bada, ikusi (1.6)–. Orobat, Jaikin-Zapirainek [43]n antzeko ideiak erabili zituen, R zero karakteristikako pro-p domeinuen gainean finituki sortutako talde R-analitiko trinkoak linealak direla frogatzeko. Kapitulu honetan azken emaitza hori orokortuko dugu, eta R zero karakteristikako pro-p domeinua denean, talde R-analitiko trinkoak linealak direla frogatuko da.

2.1 Talde p-adiko analitiko trinkoen linealtasuna

Atal honetan, talde p-adiko analitikoetarako arestian aipatutako adierazpen lineal leiala nola eraiki labur-labur deskribatuko dugu, eraikitako adierazpenaren maila kontrolatuz. Lehenik eta behin, gogora dezagun Adoren Teorema, edo hobeki esanda, Churkin [19] eta Weigelen [74] bertsio orokorrago hau (froga eta oinarrizko definizio guztiak Apendizea An topa daitezke):

Teorema 2.4. *Izan bedi $\mathfrak{L}$ heina finituko $\mathbb{Z}_p$-modulu askea den $\mathbb{Z}_p$-Lie aljebra. Orduan, existitzen da $\phi\colon \mathfrak{L} \hookrightarrow \mathrm{M}_n(\mathbb{Z}_p)$ $\mathbb{Z}_p$-Lie aljebra monomorfismoa non n maila soilik $\mathfrak{L}$-ren heinaren menpekoa den.*

Finkatu p zenbaki lehena, eta notazioa sinplifikatzearren, definitu $\mathbf{p} = 2$, $p = 2$ denean eta $\mathbf{p} = p$, p bakoitia denean. Talde $\mathbb{Z}_p$-estandar oro *talde uniformeki berretura-betea* deiturikoa ere bada; hots, S talde $\mathbb{Z}_p$-estandarra bada, finituki sortutako talde tortsiogabea da eta $S' \leq S^{\mathbf{p}}$ betetzen du (ikusi [24, Teoremak

4.5 eta 8.31]). Orobat, kategoria isomorfismo bat dago, d-dimentsioko pro-p talde uniformeki berretura-beteen $\mathfrak{UGroup}$ kategoriaren eta d-dimentsioko $\mathbb{Z}_p$-*erretikulu berretura-beteen* $\mathfrak{pLie}$ kategoriaren artean ($\mathfrak{L}$ $\mathbb{Z}_p$-Lie aljebra $\mathbb{Z}_p$-Lie erretikulu berretura-betea da, d heinako $\mathbb{Z}_p$-modulu askea bada eta $[\mathfrak{L},\mathfrak{L}] \leq \mathbf{p}\mathfrak{L}$ betetzen badu). Alegia, $\mathcal{L}\colon \mathfrak{UGroup} \to \mathfrak{pLie}$ kategoria isomorfimo bat dago, deitu $\mathcal{E}$ haren alderantzizkoari. Isomorfismo horrek d dimentsioko S talde uniformeki berretura-bete bakoitzari d dimentsioko $\mathcal{L}(S)$ $\mathbb{Z}_p$-Lie erretikulu berretura-bete bat esleitzen dio, oinarrian datzan multzoa S bera da, eta modulu eragiketen talde eragiketaren arabera honako moduan definitzen dira: izan bitez $r \in \mathbb{Z}_p$ eta $x, y \in S$, orduan

$$r \cdot x = x^r$$
$$x + y = \lim_{n\to\infty} \left(x^{p^n} y^{p^n}\right)^{p^{-n}}$$
$$[x,y] = \lim_{n\to\infty} \left(\left[x^{p^n}, y^{p^n}\right]\right)^{p^{-2n}},$$

ikusi [24, Atala 4.3] eskuineko formulen definizio zehatzetarako. Esate baterako, $\mathbf{p}\,\mathrm{M}_n(\mathbb{Z}_p)$ multzoa $\mathbb{Z}_p$-Lie aljebra berretura-betea da $[A,B] = AB - BA$ Lieren kortxetearekiko; eta adibide honetan $\mathcal{E}(\mathbf{p}\,\mathrm{M}_n(\mathbb{Z}_p)) \subseteq \mathrm{GL}_n(\mathbb{Z}_p)$ da, $\mathcal{E}$ ohiko matrize exponentziazioa balitz bezala ikus baitaiteke*.

Alderantzizko norantzan, *Baker-Hausdorff-Campbellen formula* bi aldagai eztrukakorretan definituriko $H(x,y)$ berretura serie formala da eta

$$e^{H(x,y)} = e^x e^y$$

identitatea betetzen du. Horrela, $\mathcal{S}$ $\mathbb{Z}_p$-Lie erretikulu berretura-betea bada, $\mathcal{S}$ multzoa talde uniformeki berretura-betea da $xy = H(x,y)$ talde eragiketarekin.

Ondorioz, Adoren Teoremak (Teorema 2.4) emandako $\mathbb{Z}_p$-Lie aljebra monomorfismoa $\phi\colon \mathfrak{L}(S) \hookrightarrow \mathrm{M}_n(\mathbb{Z}_p)$ bada, bada Adoren Teoremak, Teorema 2.4, emandako bada, orduan

$$\phi|_{\mathbf{p}\mathcal{L}(S)}\colon \mathbf{p}\mathcal{L}(S) \to \mathbf{p}\,\mathrm{M}_n(\mathbb{Z}_p)$$

$\mathbb{Z}_p$-Lie erretikulu berretura-beteen arteko $\mathbb{Z}_p$-Lie aljebra monomorfismoa da, eta $\mathcal{E}$ funktorea denez,

$$\mathcal{E}(\phi)\colon \mathcal{E}(\mathbf{p}\mathcal{L}(S)) \hookrightarrow \mathcal{E}(\mathbf{p}\,\mathrm{M}_n(\mathbb{Z}_p)) \subseteq \mathrm{GL}_n(\mathbb{Z}_p)$$

talde monomorfismoa lortzen dugu.

*Benetan $\mathcal{E}\left(\mathbf{p}\,\mathrm{M}_n(\mathbb{Z}_p)\right) = \mathrm{GL}_n^1(\mathbb{Z}_p)$ lehenbizko kongruentzia taldea da.

Azkenik, $\mathbb{Z}_p$-Lie erretikulu berretura-betean batuketarekiko koklaseak eta talde uniformeki berretura-betean talde eragiketarekiko koklaseak bat datoz (ikusi [24, Proposizioa 4.31(iii)]). Beraz,

$$|S : \mathcal{E}\left(\mathbf{p}\mathcal{L}(S)\right)| = |\mathcal{L}(S) : \mathcal{L} \circ \mathcal{E}(\mathbf{p}\mathcal{L}(S))| = |\mathcal{L}(S) : \mathbf{p}\mathcal{L}(S)| = \left|\mathbb{Z}_p^{(d)} : \mathbf{p}\mathbb{Z}_p^{(d)}\right| = \mathbf{p}^d.$$

Hori dela eta, adierazpen eratorria erabiliz,

$$m \colon S \hookrightarrow \mathrm{GL}_{\mathbf{p}^d n}(\mathbb{Z}_p)$$

talde monomorfismoa erdiesten dugu.

Argudio horiek guztiak batuz honako emaitza dugu:

Teorema 2.5. *Izan bedi S talde $\mathbb{Z}_p$-estandarra. Orduan, existitzen da $m \colon S \hookrightarrow \mathrm{GL}_n(\mathbb{Z}_p)$ S-ren adierazpen lineal leiala, eta ϕ-ren n maila soilik S-ren dimentsioaren eta p-ren menpekoa da.*

Bukatzeko, talde $\mathbb{Z}_p$-analitiko trinko guztiek indize finituko talde $\mathbb{Z}_p$-estandar bat dutenez (ikusi Lema 1.21) adierazpen eratorriak Lazarden jatorrizko emaitza ematen digu:

Korolarioa 2.6 (Lazard)**.** *Talde p-adiko analitiko trinkoak $\mathbb{Z}_p$-ren gainean linealak dira.*

2.2 Pro-p domeinu aldaketa

Tesi honetako hainbat emaitzaren atzean idea bera dago: problema bat Krull dimentsioko pro-p domeinu batera murriztu eta talde horietan jada ezagunak diren egitura-emaitzak erabili. Horretarako pro-p domeinuen arteko $\varphi \colon R \to Q$ homomorfismoak erabiliko dira, talde R-analitiko batetik abiatuta talde Q-anatilikoak eraikitzeko. Zehazkiago, izan bedi $F(\mathbf{X}) = \sum_{\alpha \in \mathbb{N}_0^{(d)}} a_\alpha \mathbf{X}^\alpha \in R[[\mathbf{X}]]$ berretura serie formala, koefizienteei φ aplikatuz

$$\mathbf{F}_\varphi = \sum_{\alpha \in \mathbb{N}_0^{(d)}} \varphi(a_\alpha)\mathbf{X}^\alpha \in Q[[\mathbf{X}]]$$

berretura serie formala dugu (hau berretura serie formalen propietate unibertsalaren kasu jakin bat baino ez da, ikusi Teorema 1.6).

Koefizienteak aldatzeko eraztun lokalen arteko homomorfismo naturalak bakarrik hartuko ditugu aintzat, *eraztun homomorfismo lokalak* hain zuzen ere. Horiek eraztun lokalen arteko $\varphi\colon (R,\mathfrak{m}) \to (Q,\mathfrak{n})$ homomorfismoak dira eta $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$ betetzen dute. Bide batez, ohartu eraztun homomorfismo lokalak jarraituak direla, $\varphi(\mathfrak{m}^n) \subseteq \mathfrak{n}^n$ baita $n \in \mathbb{N}$ guztietarako.

Hurrengo lemaren arabera, deskribaturiko eraztun aldaketa eta berretura serieen konposizioa trukakorrak dira.

Lema 2.7. *Izan bitez $\varphi\colon (R,\mathfrak{m}) \to (Q,\mathfrak{n})$ eraztun homomorfismo lokala eta $\boldsymbol{F} \in R[[X_1,\ldots,X_n]]^{(m)}$ eta $\mathbf{G} \in R[[X_1,\ldots,X_m]]^{(l)}$ berretura serie formalak. Demagun $\boldsymbol{F}(\mathbf{0}) \in \mathfrak{m}^{(m)}$ dela. Orduan, $(\mathbf{G}\circ\boldsymbol{F})_\varphi(X_1,\ldots,X_n) = \mathbf{G}_\varphi \circ \boldsymbol{F}_\varphi(X_1,\ldots,X_n)$ da.*

Froga. Berretura serieen propietate unibertsalaren arabera,

$$\Phi_\varphi\colon R[[X_1,\ldots,X_n]] \to Q[[X_1,\ldots,X_n]]$$

eraztun homomorfismo jarraitu bakarra existitzen da, zeinak $\Phi_\varphi(\mathbf{H}(\mathbf{X})) = \mathbf{H}_\varphi(\mathbf{X})$ betetzen duen $\mathbf{H} \in R[[\mathbf{X}]]$ guztietarako, hemen $\mathbf{X} = (X_1,\ldots,X_n)$ da.

Izan bedi $\mathbf{F}(\mathbf{X}) = (F_1(\mathbf{X}),\ldots,F_m(\mathbf{X}))$. Orduan, $\mathbf{F}(\mathbf{0}) \in \mathfrak{m}^{(m)}$ eta φ lokala direnez, $\Phi_\varphi(F_1(\mathbf{X})),\ldots,\Phi_\varphi(F_m(\mathbf{X}))$ berretura seriak $(\mathfrak{n},X_1,\ldots,X_n)Q[[\mathbf{X}]]$-n daude, hau da, $Q[[\mathbf{X}]]$-ren ideal maximalean. Horregatik, propietate unibertsalaren eraginez,

$$\Phi_1\colon R[[Y_1,\ldots,Y_m]] \to Q[[X_1,\ldots,X_n]]$$

eraztun homomorfismoa defini dezakegu $\Phi_1(r) = \varphi(r)$, $r \in R$ guztietarako, eta $\Phi_1(Y_i) = (F_i)_\varphi(\mathbf{X})$, $i \in \{1,\ldots,m\}$ guztietarako, moduan. Modu berean, $F_1(\mathbf{X})$, …, $F_m(\mathbf{X})$ berretura serieak $R[[\mathbf{X}]]$ eraztunaren ideal maximalean daudenez,

$$\Phi_2\colon R[[Y_1,\ldots,Y_m]] \to R[[X_1,\ldots,X_n]]$$

aplikazioa defini dezakegu, $\Phi_2|_R = \mathrm{Id}_R$ eta $\Phi_2(Y_i) = F_i(\mathbf{X})$, $i \in \{1,\ldots,m\}$, moduan.

Ohartu $\Phi_1(r) = \Phi_\varphi \circ \Phi_2(r) = \varphi(r)$ dela $r \in R$ guztietarako eta $\Phi_1(Y_i) = \Phi_\varphi \circ \Phi_2(Y_i) = (F_i)_\varphi(\mathbf{X})$ dela $i \in \{1,\ldots,m\}$ guztietarako. Horrenbestez, propietate unibertsalaren bakartasuna dela eta, $\Phi_1 = \Phi_\varphi \circ \Phi_2$ da. Bereziki,

$$(\mathbf{G}\circ\mathbf{F})_\varphi(\mathbf{X}) = \Phi_\varphi \circ \Phi_2(\mathbf{G}(\mathbf{Y})) = \Phi_1(\mathbf{G}(\mathbf{Y})) = \mathbf{G}_\varphi \circ \mathbf{F}_\varphi(\mathbf{X}). \qquad \square$$

Hortaz, eraztun aldaketak berretura serieen identitate formalak mantentzen ditu. Bereziki:

Korolarioa 2.8. *Izan bitez R eta Q pro-p domeinuak, $\boldsymbol{F} \in R[[X_1,\ldots,X_{2d}]]^{(d)}$ talde eragiketa formala, $\boldsymbol{I}$ haren alderantzizko formala eta $\varphi\colon R \to Q$ eraztun homomorfismo lokala. Orduan, $\boldsymbol{F}_\varphi$ talde eragiketa formala da eta haren alderantzizko formala $\boldsymbol{I}_\varphi$ da.*

Froga. Izan bitez $\mathbf{X}$, $\mathbf{Y}$ eta $\mathbf{Z}$ aldagaien d-tuplak. Ohartu $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ dela, eta, beraz, Lema 2.7 dela eta,

(i) $\mathbf{F}(\mathbf{F}(\mathbf{X},\mathbf{Y}),\mathbf{Z}) = \mathbf{F}(\mathbf{X},\mathbf{F}(\mathbf{Y},\mathbf{Z}))$ denez,

$$\mathbf{F}_\varphi(\mathbf{F}_\varphi(\mathbf{X},\mathbf{Y}),\mathbf{Z}) = \mathbf{F}_\varphi(\mathbf{X},\mathbf{F}_\varphi(\mathbf{Y},\mathbf{Z}))$$

da, eta

(ii) $\mathbf{F}(\mathbf{X},\mathbf{0}) = \mathbf{F}(\mathbf{0},\mathbf{X}) = \mathbf{X}$ denez, $\mathbf{F}_\varphi(\mathbf{X},\mathbf{0}) = \mathbf{F}_\varphi(\mathbf{0},\mathbf{X}) = \mathbf{X}$ da.

Hortaz, $\mathbf{F}_\varphi \in Q[[X_1,\ldots,X_{2d}]]^{(d)}$ talde eragiketa formala da. Azkenik, $\mathbf{I}(\mathbf{0}) = \mathbf{0}$ eta $\mathbf{F}(\mathbf{I}(\mathbf{X}),\mathbf{X}) = \mathbf{F}(\mathbf{X},\mathbf{I}(\mathbf{X})) = \mathbf{0}$ direnez, Lema 2.7 dela eta,

$$\mathbf{F}_\varphi\left(\mathbf{I}_\varphi(\mathbf{X}),\mathbf{X}\right) = \mathbf{F}_\varphi\left(\mathbf{X},\mathbf{I}_\varphi(\mathbf{X})\right) = \mathbf{0}$$

da, hau da, $\mathbf{I}_\varphi$ da $\mathbf{F}_\varphi$-ren alderantzizko formala. $\square$

Izan bitez orain $\varphi\colon (R,\mathfrak{m}) \to (Q,\mathfrak{n})$ pro-p domeinuen arteko eraztun homomorfismo lokala eta S talde R-estandarra, zeina $\left(\mathfrak{m}^N\right)^{(d)}$-rekin identifikatu daitekeen, eta demagun $\mathbf{F}$ dela S-ren talde eragiketa formala. Aitzineko $\mathbf{F}_\varphi$ talde eragiketa formala erabilita $L := \left(\mathfrak{n}^N\right)^{(d)}$ multzoa talde Q-estandar egitura batekin hornitu daiteke. Azken batean, talde eragiketa

$$x * y = \mathbf{F}_\varphi(x,y) \tag{2.2}$$

da $x, y \in L$ guztietarako. Hurrengo leman orain arteko notazioa mantenduko dugu.

Lema 2.9. *Izan bedi*

$$\varphi^{(d)}\colon \left(\left(\mathfrak{m}^N\right)^{(d)}, \boldsymbol{F}\right) \to \left(\left(\mathfrak{n}^N\right)^{(d)}, \boldsymbol{F}_\varphi\right),\ (r_1,\ldots,r_d) \mapsto (\varphi(r_1),\ldots,\varphi(r_d))$$

funtzioa. Orduan, $\varphi^{(d)}$ talde homomorfismoa da.

Froga. Idatzi $\mathbf{F} = (F_1,\ldots,F_d)$ eta

$$F_i(\mathbf{X},\mathbf{Y}) = \sum_{\alpha,\beta\in\mathbb{N}_0^{(d)}} a_{\alpha,\beta}\mathbf{X}^\alpha\mathbf{Y}^\beta \in R[[\mathbf{X},\mathbf{Y}]].$$

Orduan,

$$\begin{aligned}\varphi(F_i(x,y)) &= \varphi\left(\sum_{\alpha,\beta\in\mathbb{N}_0^{(d)}} a_{\alpha,\beta}x_1^{\alpha_1}\dots x_d^{\alpha_d}y_1^{\beta_1}\dots y_d^{\beta_d}\right)\\ &= \sum_{\alpha,\beta\in\mathbb{N}_0^{(d)}} \varphi(a_{\alpha,\beta})\varphi(x_1)^{\alpha_1}\dots\varphi(x_d)^{\alpha_d}\varphi(y_1)^{\beta_1}\dots\varphi(y_d)^{\beta_d}\\ &= (F_i)_\varphi\left(\varphi^{(d)}(x),\varphi^{(d)}(y)\right)\ \forall x,y\in\left(\mathfrak{m}^N\right)^{(d)},\end{aligned}$$

bigarren berdintzan φ jarraitua dela erabilita; eta, horrenbestez, $\varphi^{(d)}$ talde homomorfismoa da. □

Demagun orain R ez dela IND bat. Izan bitez orain G talde R-analitiko trinkoa eta $(S,\mathbf{F})$ azpitalde R-estandar normal irekia, eta demagun $g\in G$ guztietarako $c_g\colon S\to S$ konjokazio aplikazioa hertsiki R-analitikoa dela (Lema 1.23k ziurtatzen du horrelako azpitalde R-estandar baten existentzia). Talde Q-analitiko trinko bat eraiki behar dugu, zeinaren azpitalde Q-estandar normala (2.2)n eraikitako $L=\left(\left(\mathfrak{n}^N\right)^{(d)},\mathbf{F}_\varphi\right)$ taldea den. Zehazkiago, hartu S-ren ezker transbertsal bat G-n, deitu T, eta suposatu $1\in T$ dela. Notazioa erraztearren honako laburdura erabiliko dugu: $g\in G$ guztietarako $\tilde{g}$ moduan adieraziko dugu gS koklaseak T-n duen ordezkaria. Horrela, (1.3.1) dela eta –eta bertako notazio mantenduz–, $x\in tS$ eta $y\in rS$ guztietarako beren biderketa

$$\phi_{\widetilde{tr}}(xy)=A_{tr}^{\widetilde{tr}}\left(\mathbf{F}\left(\mathbf{C}_r(\phi_t(x)),\phi_r(y)\right)\right)$$

formulak ematen du. Definitu $H:=T\times L$ eta $\psi_t\colon(t,L)\to L$ homeomorfismoak, non $\psi_t(t,l)=l$ den. Aurreko formula imitatuz, $x\in(t,L)$ eta $y\in(r,L)$ emanda, haien biderketa honela definitzen da:

$$x*y=\left(\widetilde{tr},\left(A_{tr}^{\widetilde{tr}}\right)_\varphi\left(\mathbf{F}_\varphi\left(\left(\mathbf{C}_r\right)_\varphi(\psi_t(x)),\psi_r(y)\right)\right)\right). \tag{2.3}$$

Oharra. Sinpletasunagatik $(1,L)$ koklasea L-rekin identifika dezakegu, eta orduan (2.3) eragiketak (2.2) hedatzen du.

Lema 2.10. *Orain arteko notazioa mantenduz, $(H,*)$ talde Q-analitiko trinkoa da eta $L\trianglelefteq H$ azpitalde normal ireki Q-estandarra.*

Froga. Lehenik eta behin, H barietate Q-analitiko trinkoa da $\{(tL, \psi_t)\}_{t\in T}$ atlasarekiko –notazio abusu batekin, tL moduan izendatuko dugu (t, L) koklasea–. Halaber, H taldea da. Izan ere,

(i) $t, r, p \in T$ badira, Lema 1.24 eta taldearen elkarkortasunaren ondorioz, berretura serie formal gisa

$$A^{\widetilde{trp}}_{t\cdot\widetilde{rp}}\left(\mathbf{F}\left(\mathbf{C}_{\widetilde{rp}}(\mathbf{X}), A^{\widetilde{rp}}_{rp}(\mathbf{F}(\mathbf{C}_p(\mathbf{Y}),\mathbf{Z}))\right)\right) = A^{\widetilde{trp}}_{\widetilde{tr}\cdot p}\left(\mathbf{F}\left(\mathbf{C}_p\left(A^{\widetilde{tr}}_{tr}\left(\mathbf{F}\left(\mathbf{C}_r(\mathbf{X}),\mathbf{Y}\right)\right)\right),\mathbf{Z}\right)\right)$$

betetzen da. Hortaz, $x \in tL$, $y \in rL$ eta $z \in pL$ badira, Lema 2.7 eta identitatea (2.3)ren eraginez,

$$\begin{aligned}
&\psi_{\widetilde{trp}}(x * (y * z)) \\
&\quad= \left(A^{\widetilde{trp}}_{t\cdot\widetilde{rp}}\right)_\varphi \left(\mathbf{F}_\varphi\left(\left(\mathbf{C}_{\widetilde{rp}}\right)_\varphi(\psi_t(x)), \left(A^{\widetilde{rp}}_{rp}\right)_\varphi\left(\mathbf{F}_\varphi\left(\left(\mathbf{C}_p\right)_\varphi(\psi_r(y)), \psi_p(z)\right)\right)\right)\right) \\
&\quad= \left(A^{\widetilde{trp}}_{\widetilde{tr}\cdot p}\right)_\varphi \left(\mathbf{F}_\varphi\left(\left(\mathbf{C}_p\right)_\varphi\left(\left(A^{\widetilde{tr}}_{tr}\right)_\varphi\left(\mathbf{F}_\varphi\left(\left(\mathbf{C}_r\right)_\varphi(\psi_t(x)), \psi_r(y)\right)\right)\right), \psi_p(z)\right)\right) \\
&\quad= \psi_{\widetilde{trp}}((x * y) * z)
\end{aligned}$$

betetzen da.

(ii) Elementu neutroa $(1, \mathbf{0}) \in L$ da.

(iii) Izan bedi $x \in tL$, orduan alderantzizkoa

$$y = \left(\widetilde{t^{-1}}, \left(A^{\widetilde{t^{-1}}}_{t^{-1}}\right)_\varphi \circ \left(\mathbf{C}_{t^{-1}}\right)_\varphi \circ \mathbf{I}_\varphi(\psi_t(x))\right)$$

da. Izan ere, $x * y \in L$ da eta Lema 1.24 kontuan hartuta,

$$A^1_{t\cdot\widetilde{t^{-1}}}\left(\mathbf{F}\left(\mathbf{C}_{\widetilde{t^{-1}}}(\mathbf{X}), A^{\widetilde{t^{-1}}}_{t^{-1}}\left(\mathbf{C}_{t^{-1}}(\mathbf{I}(\mathbf{X}))\right)\right)\right) = \mathbf{0}$$

dugu. Hortaz, Lema 2.7 eta identitatea (2.3)ren eraginez,

$$\begin{aligned}
\mathbf{0} &= \left(A^1_{t\cdot\widetilde{t^{-1}}}\right)_\varphi \left(\mathbf{F}_\varphi\left(\left(\mathbf{C}_{\widetilde{t^{-1}}}\right)_\varphi(\psi_t(x)), \left(A^{\widetilde{t^{-1}}}_{t^{-1}}\right)_\varphi\left(\left(\mathbf{C}_{t^{-1}}\right)_\varphi\left(\mathbf{I}_\varphi(\psi_t(x))\right)\right)\right)\right) \\
&= \left(A^1_{t\cdot\widetilde{t^{-1}}}\right)_\varphi \left(\mathbf{F}_\varphi\left(\left(\mathbf{C}_{\widetilde{t^{-1}}}\right)_\varphi(\psi_t(x)), \psi_{\widetilde{t^{-1}}}(y)\right)\right) \\
&= \psi_1(x * y).
\end{aligned}$$

da. Era berean, $y * x = (1, \mathbf{0})$ da.

Azkenik, L normala da H-n, $1^t = 1$ baita $t \in T$ guztietarako. $\square$

Bukatzeko berehalakoa (baina datozen kapituluetan lagungarria) den ohar hau dugu egiteko:

Oharra 2.11. Talde Q-analitikoaren eraikuntzaren arabera,

$$\phi_{\widetilde{tr}}(x \cdot y) = \mathbf{M}\left(\phi_t(x), \phi_r(y)\right) \quad \forall (x, y) \in tS \times rS.$$

bada $\mathbf{M} \in R[[X_1, \ldots, X_{2d}]]^{(d)}$ berretura serie formal egokiarentzat, orduan,

$$\psi_{\widetilde{tr}}(\bar{x} * \bar{y}) = \mathbf{M}_\varphi\left(\psi_t(\bar{x}), \psi_r(\bar{y})\right) \quad \forall (\bar{x}, \bar{y}) \in tL \times rL$$

da. Orobat, berdina gertatzen da alderantzizko funtzioarekin.

2.2.1 Ebaluazio epimorfismoak

Praktikan aurreko pro-p domeinu aldaketa batik bat *ebaluazio epimorfismoak* erabiliz eginen dugu. Alegia, izan bitez $(R, \mathfrak{m})$ pro-p domeinua eta $a \in \mathfrak{m}^{(m)}$, a-ri dagokion ebaluazio epimorfismoa

$$s_a \colon R[[t_1, \ldots, t_m]] \to R, \; F \mapsto F(a)$$

eraztun epimorfismo lokala da. Epimorfismo horiek $R[[t_1, \ldots, t_m]]$-ren zein-nahi eraztun hedadura integraletara heda daitezke.

Teorema 2.12 (Igotze Teorema (cf. [75, Teorema V.2.3]))**.** *Izan bedi $A \subseteq B$ eraztun hedadura integrala. Orduan, edozein $\mathfrak{p} \subseteq A$ ideal lehenetarako existitzen da $\mathfrak{q} \subseteq B$ ideal lehena non $\mathfrak{q} \cap A = \mathfrak{p}$ den.*

Korolarioa 2.13. *Izan bitez $A \subseteq B$ eraztun hedadura integrala, P integritate domeinua eta $\varphi \colon A \to P$ eraztun epimorfismoa. Orduan, existitzen dira Q integritate domeinua eta φ hedatzen duen $\tilde{\varphi} \colon B \to Q$ epimorfismoa.*

Froga. Izan bedi $\mathfrak{p} = \ker \varphi$, Igotze Teoremaren arabera, existitzen da $\mathfrak{q} \subseteq B$ ideal lehena $\mathfrak{q} \cap A = \mathfrak{p}$ betez. Hori dela eta, diagrama hau trukakorra da:

$$\begin{array}{ccc} B & \xrightarrow{\tilde{\varphi}} & B/\mathfrak{q} \\ \uparrow & & \uparrow \psi \\ A & \xrightarrow{\varphi} & A/\mathfrak{p} \end{array}$$

non $\psi(x + \mathfrak{p}) = x + \mathfrak{q}$ injektiboa den. Horrela $A/\mathfrak{p}$ eraztuna P-rekin identifikatzen badugu, $\tilde{\varphi}$ epimorfismoak φ hedatzen du. $\square$

Oharra 2.14. Aurreko frogak are informazio gehiago ematen du Q-ri buruz. Izan ere, $A \subseteq B$ eraztun hedadura integrala denez, $P \subseteq Q$ ere eraztun hedadura integrala da. Izan ere, B-ko elementu baten erlazio A-integral bat modulo $\mathfrak{p}$ murriztuta, $B/\mathfrak{q}$-n dagokion elementuaren erlazio $A/\mathfrak{p}$-integral bat erdiesten da.

Are gehiago, B pro-p domeinua bada, Q ere hala da, B eraztuna ideal lehen batekin zatituz definitzen da eta.

Azkenik, teoremaren enuntziatuko $A \subseteq B$ hedadura finituki sortua bada, $P \subseteq Q$ eraztun hedadura ere finituki sortua da. Azken batean, hartu B A-modulu gisa sortzen duten elementuak eta murriztu horiek modulo $\mathfrak{q}$; elementu murriztu horiek $B/\mathfrak{q}$ sortzen dute $A/\mathfrak{p}$-modulu gisa.

Ebaluazio epimorfismo hedatu horiek pro-p domeinu guztietan ditugu eskuragarri. Izan ere, Cohenen Egitura Teoremaren (Teorem 1.2) arabera, edozein R pro-p domeinu finituki sortutako eraztun hedadura da $P[[t_1, \ldots, t_m]]$-ren gainean, $m = \dim_{\mathrm{Krull}}(R) - 1$ eta $(P, \mathfrak{n})$ bat Krull dimentsioko pro-p domeinu egoki bat izanik –benetan are gehiago esan daiteke, $P = \mathbb{Z}_p$, $\operatorname{char} R = 0$ denean, eta $P = \mathbb{F}_p[[t]]$, $\operatorname{char} R = p$ denean, baita–. Beraz, $a \in \mathfrak{n}^{(m)}$ guztietarako Q_a integritate domeinua eta $\tilde{s}_a \colon R \to Q_a$ epimorfismo jarraitua dugu. Azkenik, Lema 1.8ren antzeko propietate bat emanen dugu, paragrafo honetan zehar erabilitako notazioa mantenduz:

Korolarioa 2.15. *Izan bitez $U \subseteq_o \mathfrak{n}^{(m)}$ irekia eta $D \subseteq U$ azpimultzo dentsoa. Orduan, $\bigcap_{a \in D} \ker \tilde{s}_a = \{0\}$ da.*

Froga. Izendatu $A = P[[t_1, \ldots, t_m]]$, $\mathfrak{p}_a = \ker s_a$ eta $\mathfrak{q}_a = \ker \tilde{s}_a$ non $a \in \mathfrak{n}^{(m)}$. Lehenik eta behin, $F \in P[[t_1, \ldots, t_m]]$ berretura seriea ebaluatzea jarraitua denez, $F(a) = 0$ bada $a \in D$ guztietarako, orduan $F(a) = 0$ da $a \in U$ guztietarako. Orobat, definizioz $\mathfrak{p}_a = \mathfrak{q}_a \cap A$ da, eta, beraz, Lema 1.8ren ondorioz,

$$\left(\bigcap_{a \in D} \mathfrak{q}_a\right) \cap A = \bigcap_{a \in D} \mathfrak{p}_a = \bigcap_{a \in U} \mathfrak{p}_a = \{0\}$$

da. Demagun absurdura eramanez, existitzen dela $r \in \bigcap_{a \in D} \mathfrak{q}_a \setminus \{0\}$. Orduan, $A \subseteq R$ eraztun hedadura integrala denez, existitzen da $f_r(X) \in A[X]$ polinomio monikoa non

$$f_r(r) = r^n + \sum_{i=0}^{n-1} a_i r^i = 0$$

den. Areago, demagun f_r maila minimokoa dela. Orain, $\bigcap_{a \in D} \mathfrak{q}_a$ ideala denez, $a_0 \in \bigcap_{a \in D} \mathfrak{q}_a \cap A = \{0\}$ da, baina hori n minimoa izatearekin kontraesana da, $f_r(X) = X^{n-1} + \sum_{i=1}^{n-1} a_i X^{i-1}$ hartu baikenezakeen. □

Atal honen xedea honako kontzeptu hau aztertzea da:

Definizioa 2.16. Izan bitez A eta B egitura aljebraiko beraren bi adibide. Orduan, A *guztiz erresidualki* B dela edo B-k A diskriminatzen duela diogu, $S \subseteq A$ azpimultzo finitu guztietarako existitzen bada $h\colon A \to B$ homomorfismoa (dagokion kategorian) non $h|_S$ murrizketa injektiboa den.

Orokorrago, $\mathcal{B} = \{B_i\}_{i\in I}$ familiak A diskriminatzen du edo A guztiz erresidualki $\mathcal{B}$ da, $S \subseteq A$ azpimultzo finitu bakoitzerako existitzen bada $h\colon A \to B_i$ homomorfismo bat, $i \in I$ baten baterako, non $h|_S$ injektiboa den.

Adibidea 2.17. Eman dezagun adibide bat: izan bitez A eta $\mathcal{B} = \{B_i\}_{i\in I}$ eraztunak eta demagun A integritate domeinua dela. Orduan, A guztiz erresidualki $\mathcal{B}$ da baldin eta soilik baldin existitzen bada $\mathcal{F} \subseteq \cup_{i\in I} \operatorname{Hom}(A, B_i)$ homomorfismo familia non

$$\bigcap_{f\in\mathcal{F}} \ker f = \{0\}$$

den. *Soilik baldina* agerikoa da, bestela $x \in A$ elementu ez-nulu bat bailegoke non $f(x) = 0$ den $f\colon A \to B_i$ eraztun homomorfismo guztietarako, eta, beraz, ez legoke $\{0, x\}$ multzo finituan injektiboa den eraztun homomorfismorik.

*Baldin*a ikusteko, izan bedi $S = \{r_i\}_{i\in I} \subseteq A$ azpimultzo finitua. Definitu $r = \prod_{i\neq j}(r_i - r_j) \in R$. Ohartu $r \neq 0$ dela, r_i guztiak ezberdinak direlako eta A integritate domeinua delako. Hori dela eta, existitzen da $f \in \operatorname{Hom}(A, B_i)$ non

$$0 \neq f(r) = \prod_{i\neq j} (f(r_i) - f(r_j))\,,$$

den. Bereziki, $f(r_i) \neq f(r_j)$ da $i \neq j \in I$ guztietarako.

Esate baterako, Lema 1.8ren arabera, R pro-p domeinuak $R[[\mathbf{X}]]$ diskriminatzen du edozein aldagai kopururako $\mathbf{X}$-n.

Lema 2.18. *Izan bedi R pro-p domeinua. Orduan, edozein $S \subseteq R$ mutzo finitutarako existitzen dira bat Krull dimentsioko eta* $\operatorname{char} R$ *karakteristitako Q pro-p domeinua eta $\varphi\colon R \to Q$ eraztun epimorfismo lokala non $\varphi|_S$ injektiboa den. Hots, R pro-p domeinua bat Krull dimentsioko eta* $\operatorname{char} R$ *karakteristikako pro-p domeinen bildumak diskriminatzen du.*

Froga. Izan bedi $m = \dim_{\mathrm{Krull}}(R) - 1$ eta demagun $(P, \mathfrak{n})$ pro-p domeinua $\mathbb{Z}_p$ dela $\mathrm{char}(R) = 0$ denean, eta $P = \mathbb{F}_p[[t]]$ dela $\mathrm{char}(R) = p$ positiboa denean. Azpiatala 2.2.1 dela eta, $a \in \mathfrak{n}^{(m)}$ bakoitzerako existitzen dira Q_a pro-p domeinua eta $s_a \colon P[[t_1, \dots, t_m]] \to P$ ebaluazio epimorfismoa hedatzen duen $\tilde{s}_a \colon R \to Q_a$ eraztun epimorfimo lokala. Halaber, Oharra 2.14ren arabera, $P \subseteq Q_a$ eraztun hedadura integrala da, eta, beraz,

$$\dim_{\mathrm{Krull}} Q_a = \dim_{\mathrm{Krull}} P = 1$$

eta

$$\mathrm{char}\, Q_a = \mathrm{char}\, P = \mathrm{char}\, R$$

dira. Azkenik, Korolarioa 2.15n arabera, $\bigcap_{a \in \mathfrak{n}^{(m)}} \ker \tilde{s}_a = \{0\}$, eta, horrenbestez, Adibidea 2.17k ematen digu emaitza. $\square$

Arestiko frogan aipatu denez, R-k zero karakteristika badu, $\mathbb{Z}_p[[t_1, \dots, t_m]]$ eraztunaren hedadura finitua da. Izan bedi $\mu(R)$ zenbaki arrunta R pro-p domeinua $\mathbb{Z}_p[[t_1, \dots, t_m]]$-modulu gisa sortzeko beharrezkoa den elementu kopuru minimoa. Orduan, Oharra 2.14ren arabera, Q_a gehienez $\mu(R)$ heinako $\mathbb{Z}_p$-modulu askea da.

Proposizioa 2.19. *Izan bitez $(R, \mathfrak{m})$ zero karakteristikako pro-p domeinua eta G talde R-estandarra. Orduan, existitzen da $n \in \mathbb{N}$ zenbaki osoa, R domeinuaren, G-ren dimentsioaren eta G-ren mailaren menpekoa soilik dena, non G guztiz erresidualki $\mathrm{GL}_n(\mathbb{Z}_p)$ den.*

Froga. Definizioz, G taldea $\left(\mathfrak{m}^N\right)^{(d)}$-rekin identifika dezakegu eta talde egitura

$$\mathbf{F}(\mathbf{X}, \mathbf{Y}) = \sum_{\alpha, \beta \in \mathbb{N}_0^{(d)}} a_{\alpha,\beta} \mathbf{X}^{\alpha} \mathbf{Y}^{\beta} \in R[[\mathbf{X}, \mathbf{Y}]]$$

talde eragiketa formal batek ematen du. Izan bedi $S \subseteq \left(\mathfrak{m}^N\right)^{(d)}$ d-tuplen azpimultzo finitua, hau da

$$S = \left\{ r_i = (r_{i1}, \dots, r_{id}) \in R^{(d)} \right\}_{i \in I},$$

eta definitu $S' = \{ r_{ij} \mid i \in I,\ j \in \{1, \dots, d\} \} \subseteq R$. Lema 2.18 dela eta, existitzen dira Q_S bat Krull dimentsioko eta zero karakteristitako pro-p domeinua eta $\pi_S \colon (R, \mathfrak{m}) \to (Q_S, \mathfrak{n})$ eraztun homomorfismo lokala, zeina injektiboa den S'-ra murriztean.

Izan bedi $H_S = \left(\mathfrak{n}^N\right)^{(d)}$ taldea,

$$\mathbf{F}_{\pi_S}(\mathbf{X}) = \sum_{\alpha,\beta\in\mathbb{N}_0^{(d)}} \pi_S(a_{\alpha,\beta})\mathbf{X}^\alpha\mathbf{Y}^\beta \in Q[[\mathbf{X},\mathbf{Y}]],$$

talde eragiketa formalarekin (konparatu Korolarioa 2.8) eta barietate Q_S-analitiko egitura naturalarekin. Lema 2.9ren arabera,

$$\pi_S^{(d)}\colon G \to H_S,\ (r_1,\ldots,r_d) \mapsto (\pi_S(r_1),\ldots,\pi_S(r_d))$$

talde homorfismoa da, eta berehalakoa da ohartzea $\pi_S^{(d)}$ S-ra murriztea injektiboa dela.

Halaber, Q_S heina finituko $\mathbb{Z}_p$-modulu askea da eta $\mu'_S \leq \mu(R)$ heina du. Beraz, eskalareak murriztuz (konparatu Korolarioa 1.44), H_S talde p-adiko analitikoa da eta haren dimentsioa $\mu'_S d \leq \mu(R)d$ da. Konkretuago, $\sigma\colon Q_S \to \mathbb{Z}_p^{(\mu'_S)}$ $\mathbb{Z}_p$-modulu isomorfismoa bada, orduan

$$\left(p^N\mathbb{Z}_p\right)^{(\mu'_S)} = \sigma\left(p^N Q_S\right) \subseteq \sigma\left(\mathfrak{n}^N\right)$$

da. Hortaz, H_S azpitalde $\mathbb{Z}_p$-estandar ireki bat du, alegia

$$U_S := \left(\sigma^{(d)}\right)^{-1}\left(p^N\mathbb{Z}_p^{(\mu'_S d)}\right).$$

Are gehiago, Lema 1.20ren arabera, talde indizeak eta ideal indizeak (hau da, indizeak talde batukor gisa) bat datoz, eta, beraz,

$$|H_S : U_S| \leq \left|\mathbb{Z}_p^{(\mu'_S d)} : \left(p^N\mathbb{Z}_p\right)^{(\mu'_S d)}\right| = p^{N\mu'_S d} \leq p^{N\mu(R)d}.$$

Azkenik, Teorema 2.5en ondorioz, existitzen da $m_1\colon U_S \hookrightarrow \mathrm{GL}_{\gamma(\mu(R)d)}(\mathbb{Z}_p)$ adierazpen lineal leiala, $\gamma\colon \mathbb{N} \to \mathbb{N}$ funtzio egoki batentzat. Izan bitez, orain, $n = p^{N\mu(R)d}\gamma(\mu(R)d)$ eta $m\colon H_S \hookrightarrow \mathrm{GL}_n(\mathbb{Z}_p)$ eratorritako talde adierazpen lineal leiala. Orduan, $m \circ \pi_S^{(d)}\colon G \to \mathrm{GL}_n(\mathbb{Z}_p)$ konposaketa talde homormorfismoa da eta injektiboa da S-ra murriztean. □

Hots,

Korolarioa 2.20. *Talde R-estandar oro talde lineal batek diskriminatzen du.*

Atal honetako emaitza nagusia, baita kapitulu osokoa ere, Teorema 2.27 da. Alegia, R zero karakteristikako pro-p domeinua denean, talde R-analitiko trinko guztiek adierazpen lineal leial bat dute.

Lehenengo, problema heuristikoki aztertuko dugu, eta gero, atalean zehar, ideia hori formalizatuko dugu. Aitzineko emaitzetan oinarrituta edozein G talde R-estandarretarako era naturalean

$$G \hookrightarrow \prod_{\substack{S \subseteq G \\ S \text{ finitua}}} \mathrm{GL}_n(\mathbb{Z}_p) \cong \mathrm{GL}_n\left(\prod_{\substack{S \subseteq G \\ S \text{ finitua}}} \mathbb{Z}_p\right)$$

talde monomorfismoa dugu. Haatik, $\mathbb{Z}_p$ eraztunen biderketa kartesiar infinitua ez da integritatea domeinua, eta, horrenbestez, monomorfismo hori moldatu egin behar da. Horretarako eredu teoria erabiliko da, eta irakurleak [17] kontsulta dezake gaiaren ikuspegi orokorra zein beharrezko definiziok topatu nahi izanez gero. Froga besterik gabe liburuko erreferentziak bilbatuz eraiki daitekeen arren, tesiaren osotasunagatik kontzeptuak eta argudioak artaz aurkeztuko dira.

Definizioa 2.21. Izan bedi I multzo infinitua. Orduan, I-ren *filtro* bat $\mathcal{U} \subseteq \mathcal{P}(I)$ azpimultzo bat da non

(U1) $\varnothing \notin \mathcal{U}$ den.

(U2) $A \in \mathcal{U}$ eta $A \subseteq B$ badira, orduan $B \in \mathcal{U}$ den.

(U3) $A, B \in \mathcal{U}$ badira, orduan $A \cap B \in \mathcal{U}$ den.

Filtro bat, deitu $\mathcal{U}$, *ultrafiltro* bat aurrekoez gain propietate hau betetzen duenean:

(U4) $A \subseteq I$ guztietarako, $A \in \mathcal{U}$ da edo $I \setminus A \in \mathcal{U}$ da.

Azkenik, ultrafiltro bat *ez-nagusia* da aurrekoez gain

(U5) $\{i\} \notin \mathcal{U}$ bada $i \in I$ guztietarako.

Oharra 2.22. Filtroen propietate pare bat enuntziatuko dira, gerora argi edukitzearren:

(i) Izan bedi $\mathcal{U}$ filtroa I-n. Orduan, A edo $I \setminus A$ bietako bakarra egon daiteke $\mathcal{U}$-n. Bereziki, $\mathcal{U}$ ultrafiltroa bada, A edo $I \setminus A$ bietako bat eta bakarra dago $\mathcal{U}$-n.

(ii) Izan bedi $\mathcal{U}$ ultrafiltroa I-n eta demagun $A \cup B \in \mathcal{U}$ dela. Orduan, A edo B, bietako bat, $\mathcal{U}$-n dago. Izan ere, bestela (U4) dela eta, $I \setminus A$, $I \setminus B \in \mathcal{U}$ litzateke, eta, beraz, (U3) eta de Morganen legea direla eta,

$$I \setminus (A \cup B) = (I \setminus A) \cap (I \setminus B) \in \mathcal{U},$$

(i) puntuarekin kontraesana dena.

Ultrafiltro Teoremaren (ikusi [17, Korolarioa 1.4.4]) arabera, $\mathcal{F} \subseteq \mathcal{P}(I)$ familiak *ebakidura finituen propietatea* betetzen badu, hau da,

$$F_1 \cap \cdots \cap F_n \neq \varnothing \ \forall n \in \mathbb{N},\ \forall F_1, \ldots, F_n \in \mathcal{F}$$

bada, orduan $\mathcal{F}$ familia I-ren ultrafiltro batera heda daiteke. Hori dela eta, zeinnahi G multzo infinitu emanda, erraza da

$$\mathcal{P}_{\text{fin}}(G) := \{S \subseteq G \mid S \text{ finitua}\} \subseteq \mathcal{P}(G)$$

multzoa ultrafiltro batez hornitzea. Azken batean, $S \in \mathcal{P}_{\text{fin}}(G)$ bakoitzerako definitu $A_S = \{T \in \mathcal{P}_{\text{fin}}(G) \mid S \subseteq T\}$, eta ohartu $S, T \in \mathcal{P}_{\text{fin}}(G)$ direnean

$$A_S \cap A_T = A_{T \cup S}$$

dela. Alegia $\mathcal{V} = \{A_S \mid S \in \mathcal{P}_{\text{fin}}(G)\}$ familiak ebakidura finituen propietatea betetzen du. Ondorioz, Ultrafiltro Teoremagatik existitzen da $\mathcal{V}$ hedatzen duen $\mathcal{U}$ ultrafiltroa. Honi $\mathcal{P}_{\text{fin}}$-eko ultrafiltro natural deituko diogu. Ultrafiltro hori ez-nagusia da. Izan ere, $S \subsetneq G$ multzo finitua emanda, hartu $g \in G \setminus S$, orduan

$$A_{\{g\}} \subseteq \mathcal{P}_{\text{fin}}(G) \setminus \{S\}.$$

Beraz, (U2)ren arabera, $\mathcal{P}_{\text{fin}}(G) \setminus \{S\} \in \mathcal{U}$ da eta Oharra 2.22(i)en ondorioz, $\{S\} \notin \mathcal{U}$ da.

Definizioa 2.23. Izan bitez I multzo infinitua, $\mathcal{G} = \{G_i\}_{i \in I}$ familia eta $\mathcal{U}$ ultrafiltroa I-n. Orduan, $\mathcal{G}$ familiaren *ultrabiderketa* $\prod_{i \in I} G_i$ biderketa kartesiarra modulo

$$(g_i)_{i \in I} \sim (h_i)_{i \in I} \iff \{i \in I \mid g_i = h_i\} \in \mathcal{U}$$

baliokidetasun erlazioa da. Egitura hori $\prod_{i \in I} G_i/_{\mathcal{U}}$ moduan izendatuko dugu, eta $(g_i \mid i \in I)$ tuplaren baliokidetasun klasea $[g_i \mid i \in I]_{\mathcal{U}}$ moduan adieraziko da.

Era berean, $G_i = G$ denean $i \in I$ guztieratarako, ultrabiderketa horri G-ren *ultraberretura* deitzen zaio, eta $G^{\mathcal{U}} = \prod_{i\in I} G/\mathcal{U}$ moduan adierazten da.

Talde familia baten ultrabiderketa (edo beste edozein egitura aljebraikorena) berriro talde bat (edo dagokion egitura aljebraikoa) da biderketa osagaiz osagai egin eta baliokidetasun klasea hartuta. Hori zuzenean oinarrizko definizioak erabiliz froga daiteke, edota Łoś́en Teorematik ondoriozta daiteke.

Lema 2.24. *Izan bedi $\mathcal{U}$ ultrafiltroa I multzo infinituan.*

(i) *Demagun $\{G_i\}_{i\in I}$ talde familia dela, orduan $\bar{G} := \prod_{i\in I} G_i/\mathcal{U}$ taldea da.*

(ii) *Demagun $\{R_i\}_{i\in I}$ eraztun familia dela, orduan $\bar{R} := \prod_{i\in I} R_i/\mathcal{U}$ eraztuna da. Orobat, R_i guztiak integritate domeinuak badira, $\overline{R}$ ere integritate domeinua da.*

Froga. (i) Ohartu $\bar{G}$ multzoa $\prod_{i\in I} G_i$ taldearen zatidura baino ez dela

$$\mathcal{N} = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} G_i \middle| \{i \in I \mid x_i = 1\} \in \mathcal{U} \right\}$$

azpitalde normalarekiko. Frogatu behar dugun bakarra $\mathcal{N}$ azpitalde normala dela da. Horretarako, laburdura hauek erabiliko ditugu: $x = (x_i \mid i \in I)$ tuplarako, J_x bidez izendatuko dugu $\{i \in I \mid x_i = 1\} \subseteq I$ multzoa. Hartu $x = (x_i \mid i \in I)$ eta $y = (y_i \mid i \in I) \in \mathcal{N}$, hau da, $J_x, J_y \in \mathcal{U}$. Orduan,

$$J_x \cap J_y = J_x \cap J_{y^{-1}} \subseteq J_{xy^{-1}},$$

beraz, (U2) eta (U3) direla eta, $J_{xy^{-1}} \in \mathcal{U}$ da, eta, ondorioz, $xy^{-1} \in \mathcal{N}$ da. Antzeko moduan, $g \in \prod_{i\in I} G_i$ guztietarako $J_x \subseteq J_{x^g}$ dugu, eta, beraz, $\mathcal{N}$ azpitalde normala da.

(ii) Lehenbiziko zatia, $\bar{R}$ eraztuna dela, (i) bezala frogatzen da, hau da, $\bar{R}$ multzoa $\prod_{i\in I} R_i$-ren zatidura da

$$\mathcal{I} := \left\{ r \in \prod_{i\in I} R_i \middle| \{i \in I \mid r_i = 0\} \in \mathcal{U} \right\}$$

idealarekiko. Bigarren zatirako, $\bar{R} \neq \{0\}$ da eta, beraz, R_i guztiak integritate domeinuak direnean $\mathcal{I}$ ideal lehena dela frogatu behar da. Hartu $r = [r_i \mid i \in I]_{\mathcal{U}}, s = [s_i \mid i \in I]_{\mathcal{U}} \in \bar{R}$ non

$$[r_i s_i \mid i \in I]_{\mathcal{U}} = rs \in \mathcal{I}$$

den, hau da, $J := \{i \in I \mid r_i s_i = 0\} \in \mathcal{U}$. Definitu $J_r = \{i \in I \mid r_i = 0\}$ eta $J_s = \{i \in I \mid s_i = 0\}$. Orduan, R_i bakoitza integritate domeinua denez, $J = J_r \cup J_s$ da, eta, Oharra 2.22(ii)ren arabera, $J_r \in \mathcal{U}$ da edo $J_s \in \mathcal{U}$ da. Hots, $r \in \mathcal{I}$ da edo $s \in \mathcal{I}$ da. □

Garapen honen ostean, baditugu hasierako argudio heuristikoa moldatzeko tresna guztiak:

Proposizioa 2.25. *Talde R-estandar oro* $\mathrm{GL}_n(\mathbb{Z}_p)$*-ren ultraberretura batean murgildu daiteke.*

Froga. Proposizioa 2.19ren arabera, badago $n \in \mathbb{N}$ non hau betetzen den: $S \subseteq G$ azpimultzo finitu guztietarako, existitzen da $h_S \colon G \to \mathrm{GL}_n(\mathbb{Z}_p)$ talde homomorfismoa eta h_S-ren murrizketa S-ra funtzio injektiboa da.

Izan bedi $\mathcal{U}$ ultrafiltro naturala $\mathcal{P}_{\mathrm{fin}}(G)$-n eta definitu

$$h \colon G \to \mathrm{GL}_n(\mathbb{Z}_p)^{\mathcal{U}},\ g \mapsto [h_S(g) \mid S \in \mathcal{P}_{\mathrm{fin}}(G)]_{\mathcal{U}}$$

talde homomorfismoa.

Injektiboa dela frogatu behar dugu, hau da, $\ker h = \{1\}$ dela. Hartu $g \neq 1$, orduan $h_S(g) \neq 1$ da $S \in A_{\{1,g\}}$ guztietarako. Beraz,

$$Y := \{S \in \mathcal{P}_{\mathrm{fin}}(G) \mid h_S(g) = 1\} \subseteq \mathcal{P}_{\mathrm{fin}}(G) \setminus A_{\{1,g\}}$$

da, eta Oharra 2.22(i) dela eta, $Y \notin \mathcal{U}$ da. Hots, $h(g) \neq 1$ da eta ondorioz h talde monomorfismoa da. □

Honakoa ere beharko dugu:

Lema 2.26. *Izan bitez* $\{R_i\}_{i \in I}$ *eraztun familia eta* $\mathcal{U}$ *ultrafiltroa* I*-n. Orduan, talde gisa*

$$\prod_{i \in I} \mathrm{GL}_n(R_i)/\mathcal{U} \cong \mathrm{GL}_n\left(\prod_{i \in I} R_i/\mathcal{U}\right)$$

da.

Froga. Izan bedi

$$\begin{aligned} f \colon \prod_{i \in I} \mathrm{GL}_n(R_i)/\mathcal{U} &\longrightarrow \mathrm{GL}_n\left(\prod_{i \in I} R_i/\mathcal{U}\right) \\ \left[A^i = (a^i_{j,k})_{j,k} \mid i \in I\right]_{\mathcal{U}} &\longmapsto \left(\left[a^i_{jk} \mid i \in I\right]_{\mathcal{U}}\right)_{j,k} \end{aligned}$$

aplikazioa. Ariketa erraza da f talde isomorfismoa dela ikustea. □

Azkenik, osagai guztiak batuko ditugu emaitza nagusia frogatzeko.

Teorema 2.27. *Izan bedi R zero karakteristikako pro-p domeinua. Orduan, talde R-analitiko trinkoak linealak dira.*

Froga. Talde R-analitiko trinko orok indize finituko azpitalde R-estandar bat du (konparatu Lema 1.21), eta, horrenbestez, adierazpen eratorria dela eta, orokortasunatik galdu bage G talde R-estandarra dela suposa dezakegu.

Bestetik, Teorema 2.25en arabera, G taldea $\mathrm{GL}_n(\mathbb{Z}_p)^{\mathcal{U}}$ ultraberretura batean murgiltzen da, $n \in \mathbb{N}$ egoki baterako. Hots, Lema 2.26ren arabera,

$$G \hookrightarrow \mathrm{GL}_n(\mathbb{Z}_p)^{\mathcal{U}} \cong \mathrm{GL}_n\left(\mathbb{Z}_p^{\mathcal{U}}\right)$$

da, eta Lema 2.24(ii)ren arabera, $\mathbb{Z}_p^{\mathcal{U}}$ integritate domeinua denez, G lineala da. □

2.5 Oharrak

Atala 2.1eko frogen ideiak Lazardi zor dizkiogu. Hala ere, hemengo aurkezpena bestelakoa da, eta batik bat [24, Atala 7.3] darraio. Atala 2.4ko emaitza eta kontzeptu gehienak oinarrizkoak dira eredu teorian dihardutenentzat. Haatik, osotasunaren mesedetan xeheki azaldu ditugu. Ikus daitekeenez, eredu teoriak bere esparrutik at duen aplikazioetako bat talde batzuk linealak direla frogatzea da, eta hori oro har ez da lan erraza. Erabilera hori lehenengoz Mal'cevi otu zitzaion.

Gainerako materiala originala da, eta Casals-Ruizekin batera eginda dago. Emaitza horiek nagusiki [16]n daude, nahiz eta zati batzuk [76]n egon.

3

Hausdorffen dimentsioa talde R-analitiko trinkoetan

DIMENTSIO FRAKTALEK dimentsio topologiko nozioa orokortzen dute. Halere, dimentsio fraktal anitz daude (nahiz eta guztiak espazioaren neurketa gisako batean oinarritu), eta horien artean garrantzitsuenetarikoak Hausdorffen dimentsioa eta Minkowski-Bouliganden dimentsioa, kutxa-dimentsioa ere deiturikoa, dira.

Aipaturiko bi dimentsio horiek edozein espazio metrikotan definitu daitezke, eta talde profinituen testuinguruan Hausdorffen dimentsioa asko aztertu izan da azken hamarkadetan. Izan ere, G oinarri kontagarriko talde profinitu infinitua bada, $\{G_n\}_{n\in\mathbb{N}}$ *filtrazio serie* bat existitzen da, alegia, azpitalde irekien segida beherakor bat, identitatearen ingurune-oinarria ere badena, hau da, $\bigcap_{n\in\mathbb{N}} G_n = \{1\}$ betetzen du. Filtrazio serie horrek

$$d(x,y) = \inf\left\{|G:G_n|^{-1} \mid xy^{-1} \in G_n\right\}.$$

metrika definitzen du G-n. Distantzia horrekin G espazio metrikoa da, eta, ondorioz, bertan Hausdorffen eta Minkowski-Bouliganden dimentsioak defini daitezke (ikusi Atala 3.1 definizio zehatzetarako), hurrenez hurren, hdim eta lbdim izendatuko ditugunak. Talde profinitu batean neurri bakarra dago, Haarren neurria, baina metrika ez-baliokide anitz daude; eta eskuarki dimentsio fraktalak metrikaren –edo baliokideki metrika definitzeko erabilitako filtrazio seriearen– menpekoak dira. Filtrazio serie jakin baterako $\mathrm{hdim}_{\{G_n\}}(H)$ balioen bilduma aztertu dezakegu, H azpitalde itxietan zehar doalarik, hau da,

$$\mathrm{hspec}_{\{G_n\}}(G) := \left\{\mathrm{hdim}_{\{G_n\}}(H) \mid H \leq_c G\right\},$$

$\{G_n\}_{n\in\mathbb{N}}$ filtrazio seriearekiko G-ren *Hausdorffen espektroa* deituko duguna. Minkowski-Bouliganden espektroa modu berean definitu genezakeen arren, filtrazio serie gehienetarako $H \leq_c G$ denean $\mathrm{hdim}(H) = \mathrm{lbdim}(H)$ dela ikusiko dugu (ikusi Teorema 3.7). Horrenbestez, terminologia klasikoa jarraituz, "Hausdorff" izena erabiliko da espektroa izendatzeko.

Filtrazio seriea aldatuz gero, espektro horiek ez dute zertan elkarren antzik eduki. Esate baterako, azter dezagun $\mathbb{Z}_p \oplus \mathbb{Z}_p$ pro-p talde batukorra. Horren gisako finituki sortutako G pro-p taldeetarako badago filtrazio serie natural bat, $G_n = G^{p^n}$ gisa definituriko *p-berretura filtrazio seriea* hain zuzen ere. Filtrazio serie horrekin, $\mathrm{hspec}_{\{G_n\}}(\mathbb{Z}_p \oplus \mathbb{Z}_p) = \{0, 1/2, 1\}$ da, eta bereziki finitua da; baina, [48, Teorema 1.3]n frogatzen denez, existitzen da $\{H_n\}_{n\in\mathbb{N}}$ filtrazio seriea zeinetarako $\left[\frac{1}{p+1}, \frac{p-1}{p+1}\right] \subseteq \mathrm{hspec}_{\{H_n\}}(\mathbb{Z}_p \oplus \mathbb{Z}_p)$ den, eta, $p > 2$ denean, espektroa infinitu ezkontagarria da. Alegia, espektroaren finitutasuna ere ez da filtrazio seriearekiko aldagaitza.

Talde profinituetan Hausdorffen dimentsioa aztertzen duen artikulu goiztiarrenetakoa [6] da. Bertan, beste emaitza askoren artean, G pro-p talde p-adiko analitikoetarako $\mathrm{hspec}_{\{G^{p^n}\}}(G)$ finitua dela ikusten da (ikusi [6, Korolarioa 1.2]). Haatik, emaitza horren alderantzizkoa galdera irekia da oraindik.

Galdera 3.1 (cf. [6, Problema 1])**.** Izan bedi G finituki sortutako pro-p taldea eta demagun $\mathrm{hspec}_{\{G^{p^n}\}}(G)$ finitua dela. Orduan, G talde p-adiko analitikoa da?

Aipatu behar da aurreko G taldea ebazgarria ere badenean, galderak erantzun positiboa duela (ikusi [48, Teorema 1.7]).

Talde R-analitiko trinkoetan normalean ezin da p-berretura filtrazio seriea erabili, G^{p^n} ez baita oro har azpitalde irekia. Halere, talde horiek badute filtrazio serie kanoniko bat, taldearen egitura analitikoa aintzat hartzen duena. Etsenplu gisa, Atala 3.2n azpibarietate itxien dimentsio analitikoa eta dimentsio fraktalak, aipatutako filtrazio serie natural horrekiko definituta, erkatuko ditugu, eta identitate hau frogatuko da:

$$\mathrm{hdim}(M) = \mathrm{lbdim}(M) = \frac{\max\{\dim_x M \mid x \in M\}}{\dim H}. \tag{3.1}$$

Filtrazio serie naturalarekiko Hausdorffen dimentsioa –Azpiatala 3.1.3n aurkeztuko da sakonki– *Hausdorffen dimentsio R-estandarra* deituko dugu, eta haren Hausdorffen espektroa *Hausdoffen espektro R-estandarra*, $\mathrm{hspec}_{\mathrm{st}}$, izanen da.

Pro-p talde p-adiko analitikoetarako, Hausdorffen espektroaren finitutasuna birformulatu daiteke:

Teorema 3.2 (cf. [6, Korolarioa 1.2] eta [27, Korolarioa 3.4])**.** *Izan bedi G talde p-adiko analitiko trinkoa. Orduan,* $\mathrm{hspec}_{\mathrm{st}}(G)$ *finitua eta arrazionala da.*

Aitzitik, egoera guztiz bestelakoa da oinarriko eraztuna ez denean $\mathbb{Z}_p$-ren finituki sortutako eraztun hedadura (oroitu talde R-analitiko bat talde p-adiko analitikoa dela baldin eta soilik baldin R finituki sortutako $\mathbb{Z}_p$-ren eraztun hedadura bada). Kapitulu honetan, batez ere $R = \mathbb{F}_p[[t]]$ kasura mugatuko gara, eta honakoak frogatuko ditugu:

Teorema 3.3. *Izan bedi G talde $\mathbb{F}_p[[t]]$-analitiko trinkoa. Orduan,*

(i) *G-ren Hausdorffen espektro estandarrak $[0, {}^1/_{\dim G}]$ tarte erreala barruan du.*

(ii) *G ebazgarria bada,* $\mathrm{hspec}_{\mathrm{st}}(G) = [0, 1]$ *da.*

Emaitza horiek iradokitzen dute espektro estandarra nahikoa dela talde p-adiko analitikoak bakantzeko. Izan ere, arestiko emaitzak aieru orokor honekin bateragarriak dira:

Aierua 3.4. Izan bedi G talde R-analitiko trinkoa. Demagun $\mathrm{hspec}_{\mathrm{st}}(G)$ finitua dela. Orduan, G talde p-adiko analitikoa da.

3.1 Hausdorff eta kutxa-dimentsioak

Atal honetan, arestian aipatutako dimentsio fraktalak laburki deskribatuko ditugu. Beren propietate nagusiak eta zenbait oinarrizko emaitza bilduko ditugu, bereziki talde profinituen testuinguruan garrantzia dutenak.

3.1.1 Oinarrizko definizio eta propietateak

Has gaitezen definizio formalak emanez. Izan bitez (M, d) espazio metrikoa, $X \subseteq M$ azpimultzoa, eta δ eta z zenbaki erreal positiboak. Definitu

$$\mathcal{H}^z_\delta(X) := \inf \sum_{n=1}^{\infty} \mathrm{diam}(U_n)^z,$$

non $\{U_n\}_{n\in\mathbb{N}}$ multzoa X-ren *δ-estalkia* den, hau da, azpimultzo guztiek gehienez δ diametroa dute, eta infimoa estalki horien guztien artean hartzen da. Bestetik,

$$\mathcal{H}^z(X) := \lim_{\delta\to 0^+} \mathcal{H}^z_\delta(X)$$

limitea existitu egiten da, $\mathcal{H}^z_\delta(X)$ ez-beherakorra baita δ zerora doan heinean. Are gehiago, $\mathcal{H}^z$ kanpo-neurria da M-n (ikusi [28, Proposizioa 11.17]), *z-Hausdorffen neurria* deitu ohi dena. Honakoa betetzen da:

Lema 3.5 (cf. [26, Atala 3.2])**.** *Demagun $\mathcal{H}^s(X) < \infty$ eta $t \geq s$ direla. Orduan, $\mathcal{H}^t(X) = 0$ da.*

Horrela, $X \subseteq M$ multzoaren *Hausdorffen dimentsioa* d metrikarekiko honela definitzen da:

$$\mathrm{hdim}_d(X) := \inf\{s \mid \mathcal{H}^s(X) = 0\} = \sup\{s \mid \mathcal{H}^s(X) = \infty\}$$

–talde profinituetan, metrika $\{G_n\}_{n\in\mathbb{N}}$ filtrazio serie baten menpekoa da, eta, horrenbestez, $\mathrm{hdim}_{\{G_n\}}$ notazioa erabiliko dugu–.

Hausdorffen dimentsioa

- *monotonoa* da, hau da, $\mathrm{hdim}_d(X) \leq \mathrm{hdim}_d(Y)$ da $X \subseteq Y$ bada eta
- *kontagarri-egonkorra* da, hau da, $\mathrm{hdim}_d(\cup_{n\in\mathbb{N}} X_n) = \sup_{n\in\mathbb{N}} \mathrm{hdim}_d(X_n)$ da

(konparatu [26, 48-49 orriak]). Bestetik, propietate hau azpimarratu behar dugu:

Proposizioa 3.6 (cf. [26, Proposizioa 3.3])**.** *Izan bedi $f\colon (M_1, d_1) \to (M_2, d_2)$ espazio metrikoen arteko* funtzio bilipschitziarra*, hau da, badaude bi konstante erreal $C, c \in \mathbb{R}_{\geq 0}$ non*

$$c \cdot d_1(x,y) \leq d_2(f(x), f(y)) \leq C \cdot d_1(x,y)\ \forall x,y \in M_1$$

den. Orduan, $\mathrm{hdim}_{d_2}(f(X)) = \mathrm{hdim}_{d_1}(X)$ *da $X \subseteq M_1$ guztietarako. Bereziki, isometriek Hausdorffen dimentsioa gordetzen dute.*

Minkowski-Bouliganden dimentsioa edo *kutxa-dimentsioa* da landuko dugun beste dimentsioa. Izan bedi $N_\delta(X)$ zenbakia X multzoa estaltzeko beharrezkoa den gehienez δ diametroko azpimultzo kopuru minimoa, eta definitu, hurrenez hurren, *behe kutxa-dimentsioa* eta *goi kutxa dimentsioa* (batik bat lehenengoarekin eginen dugu lan), modu honetan:

$$\text{lbdim}_d(X) := \liminf_{\delta\to 0^+} \frac{\log N_\delta(X)}{-\log\delta} \text{ eta } \text{ubdim}_d(X) := \limsup_{\delta\to 0^+} \frac{\log N_\delta(X)}{-\log\delta}.$$

Ohartu aurreko bi adierazpenak logaritmoaren oinarriarekiko independenteak direla. Aurreko bi balioak bat datozenean, alegia zenbaki errealen segida konbergentea denean, balio komun hori X-ren kutxa-dimentsioa da, $\text{bdim}(X)$ adieraziko duguna.

Oinarri kontagarriko talde profinituetan, aitzineko limiteak soilik talde teoriako terminoak erabilita berformulatu daitezke. Horrela, G taldean $\{G_n\}_{n\in\mathbb{N}}$ filtrazio serieak definitzen duen metrikak hartzen dituen balio bakarrak $|G:G_n|^{-1}$ dira; eta, xG_n koklasea, hain zuzen ere, x zentroko eta $\delta = |G:G_n|^{-1}$ erradioko bola da. Beraz, edozein $X \subseteq G$ multzotarako $N_\delta(X) = |XG_n : G_n|$ da (honek xG_n motako koklase kopurua adierazten du $x \in X$ elementuren batentzat). Hots,

$$\text{lbdim}_{\{G_n\}}(X) = \liminf_{n\to\infty} \frac{\log|XG_n : G_n|}{\log|G:G_n|}$$

eta

$$\text{ubdim}_{\{G_n\}}(X) = \limsup_{n\to\infty} \frac{\log|XG_n : G_n|}{\log|G:G_n|}$$

dira –kasu honetan azpi-indizean $\{G_n\}$ idatziko da, metrika filtrazio serieak zehazten baitu-. Halaber, definiziotik ondorioztatzen da kutxa-dimentsioak monotonoak eta bilipschitz egonkorrak direla (ikusi [26, Proposizioa 2.5]). Gainera, goi kutxa-dimentsioa *finitu egonkorra* da, hau da,

$$\text{ubdim}(X \cup Y) = \max\{\text{ubdim}(X), \text{ubdim}(Y)\}.$$

Behe kutxa-dimentsioa, berriz, ez da beti finituki egonkorra (ikusi [26, Atala 2.2]).

Abercombriek [1] azpitalde itxietarako –eta filtrazio serie natural batzuk– Hausdorffen dimentsioa eta kutxa-dimentsioa bat datozela frogatu zuen:

Teorema 3.7 (cf. [6, Teorema 2.4])**.** *Izan bitez G oinarri kontagarriko talde profinitua eta* $\{G_n\}_{n\in\mathbb{N}}$ filtrazio serie normala*, hau da, $G_n \trianglelefteq G$ da $n \in \mathbb{N}$ guztietarako. Orduan, edozein $H \leq_c G$ azpitalde itxitarako honakoa dugu:*

$$\text{hdim}_{\{G_n\}}(H) = \text{lbdim}_{\{G_n\}}(H) = \liminf_{n\to\infty} \frac{\log|HG_n : G_n|}{\log|G:G_n|}. \tag{3.2}$$

Benetan, hasieran emandako [1] erreferentzian autoreak $\mathrm{lbdim}(H) \leq \mathrm{hdim}(H)$ dela frogatu zuen; beste ezberdintza espazio metriko guztietan betetzen da. Batik bat azpitalde itxien dimentsioaz arduratuko garenez, eta literatura matematikoan erabiltzen den terminologia mantenduz, tesian zehar gehienbat "Hausdorffen dimentsio" izena erabiliko dugu.

3.1.2 Formulak: azpitaldeak eta zatidurak

Kapituluan zehar oinarri kontagarriko talde profinitu baten Hausdorffen dimentsioa azpitaldeen eta zatidura taldeen dimentsioarekin erlazionatzea ezinbestekoa izanen da. Horregatik, zenbaitentan lagungarria izanen da $\mathrm{hdim}^G_{\{G_n\}}$ notazioa erabiltzea, dimentsioa $\{G_n\}_{n\in\mathbb{N}}$ filtrazio seriearekiko G *taldean* kalkulatzen ari garela nabarmentzeko.

Lema 3.8 (cf. [48, Lema 5.3])**.** *Izan bitez G oinarri kontagarriko talde profinitua, $\{G_n\}_{n\in\mathbb{N}}$ filtrazio serie normala G-n eta $H \leq_c G$ azpitalde itxia. Demagun H-ren Hausdorffen dimentsioa limite propioa dela. Orduan*

$$\mathrm{hdim}^G_{\{G_n\}}(K) = \mathrm{hdim}^G_{\{G_n\}}(H)\,\mathrm{hdim}^H_{\{H\cap G_n\}}(K)$$

da $K \leq_c H$ azpitalde itxi guztietarako.

Oharra 3.9. Enuntziatuan H-ren Hausdorffen dimentsioa limite propioa izanteak

$$\mathrm{hdim}_{\{G_n\}}(H) = \lim_{n\to\infty} \frac{\log|HG_n : G_n|}{\log|G : G_n|}$$

dela esan nahi du.

Froga. Kalkulu sinple bat da:

$$\begin{aligned}\mathrm{hdim}^G_{\{G_n\}}(K) &= \liminf_{n\to\infty} \frac{\log|K : K\cap G_n|}{\log|G : G_n|}\\ &= \lim_{n\to\infty} \frac{\log|H : H\cap G_n|}{\log|G : G_n|} \liminf_{n\to\infty} \frac{\log|K : K\cap H\cap G_n|}{\log|H : H\cap G_n|}\\ &= \mathrm{hdim}^G_{\{G_n\}}(H)\,\mathrm{hdim}^H_{\{H\cap G_n\}}(K).\end{aligned}$$

□

Bestetik, talde profinituen zatiduretarako hau dugu:

Lema 3.10 (cf. [47, Lema 2.2])**.** *Izan bitez G oinarri kontagarriko talde profinitua, $\{G_n\}_{n\in\mathbb{N}}$ filtrazio serie normala G-n eta $N \trianglelefteq G$ azpitalde normal itxia. Demagun N-ren Hausdorffen dimentsioa limite propioa dela. Orduan, N barnean duen edozein $H \leq_c G$ azpitalde itxitarako*

$$\mathrm{hdim}^G_{\{G_n\}}(H) = \big(1 - \mathrm{hdim}^G_{\{G_n\}}(N)\big)\,\mathrm{hdim}^{G/N}_{\{G_nN/N\}}(H/N) + \mathrm{hdim}^G_{\{G_n\}}(N).$$

Froga. Ohartu

$$\begin{aligned}
\frac{\log|HG_n:NG_n|}{\log|G:G_n|} &= \frac{\log|G:NG_n|}{\log|G:G_n|}\cdot\frac{\log|HG_n:NG_n|}{\log|G:NG_n|}\\
&= \frac{\log|G:G_n|-\log|NG_n:G_n|}{\log|G:G_n|}\cdot\frac{\log|HG_n:NG_n|}{\log|G:NG_n|}\\
&= \left(1-\frac{\log|NG_n:G_n|}{\log|G:G_n|}\right)\frac{\log|HG_n:NG_n|}{\log|G:NG_n|}.
\end{aligned}$$

Beraz, $\operatorname{hdim}^G_{\{G_n\}}(N)=\eta$ limite propioa denez,

$$\begin{aligned}
\operatorname{hdim}^G_{\{G_n\}}(H) &= \liminf_{n\to\infty}\frac{\log|HG_n:G_n|}{\log|G:G_n|}\\
&= \liminf_{n\to\infty}\left(\frac{\log|HG_n:NG_n|}{\log|G:G_n|}+\frac{\log|NG_n:G_n|}{\log|G:G_n|}\right)\\
&= \liminf_{n\to\infty}\left(\left(1-\frac{\log|NG_n:G_n|}{\log|G:G_n|}\right)\frac{\log|HG_n:NG_n|}{\log|G:NG_n|}\right)+\eta\\
&= (1-\eta)\liminf_{n\to\infty}\left(\frac{\log|HG_n/N:NG_n/N|}{\log|G/N:NG_n/N|}\right)+\eta\\
&= (1-\eta)\operatorname{hdim}^{G/N}_{\{NG_n/N\}}(H/N)+\eta,
\end{aligned}$$

nahi genuen moduan. □

Korolarioa 3.11. *Izan bitez G oinarri kontagarriko talde profinitua, $\{G_n\}_{n\in\mathbb{N}}$ filtrazio serie normala G-n eta $N\trianglelefteq G$ azpitalde normal finitua. Orduan,*

$$\operatorname{hspec}_{\{G_n\}}(G)=\operatorname{hspec}_{\{G_nN/N\}}(G/N)$$

da.

Froga. Alde batetik $\operatorname{hdim}^G_{\{G_n\}}(N)=0$ limite propioa denez,

$$\operatorname{hspec}_{\{G_nN/N\}}(G/N)\subseteq\operatorname{hspec}_{\{G_n\}}(G)$$

partekotasuna Korrespondentzia Teoremaren eta Lema 3.10en ondorio zuzena da.

Bestalde, ohartu $\eta\in\operatorname{hspec}_{\{G_n\}}(G)$ guztietarako badagoela $H\leq_c G$ azpitalde itxia non $\operatorname{hdim}^G_{\{G_n\}}(H)=\eta$ den. Hortaz, N finitua eta eskuin biderketa isomorfismoa direnez, Lema 3.10 dela eta,

$$\begin{aligned}\operatorname{hdim}^G_{\{G_n\}}(H) = \operatorname{hdim}^G_{\{G_n\}}\left(\bigcup_{n\in N} Hn\right) \\ = \operatorname{hdim}^G_{\{G_n\}}(HN) = \operatorname{hdim}^G_{\{G_nN/N\}}(HN/N)\end{aligned}$$

dugu. □

Azkenik, aurreko emaitzak konbinatuz korolario hau dugu:

Korolarioa 3.12. *Izan bitez G oinarri kontagarriko talde profinitua, $\{G_n\}_{n\in\mathbb{N}}$ filtrazio serie normala G-n eta $N \trianglelefteq K \leq G$ azpitalde itxiak. Suposa dezagun $\operatorname{hdim}^G_{\{G_n\}}(N) = \eta$ eta $\operatorname{hdim}^G_{\{G_n\}}(K) = \kappa$ dimentsioak limite propioak direla eta $\operatorname{hspec}_{\left\{\frac{(K\cap G_n)N}{N}\right\}}(K/N) = [0,1]$ dela. Orduan, $[\eta,\kappa] \subseteq \operatorname{hspec}_{\{G_n\}}(G)$ da.*

Froga. Lehenik eta behin, Lema 3.8ren ondorioz, $\operatorname{hdim}^K_{\{K\cap G_n\}}(N) = \eta/\kappa$ da, eta Korrespondentzia Teorema eta Lema 3.10 kontuan hartuta

$$[\eta/\kappa, 1] = \left\{(1-\eta/\kappa)\alpha + \eta/\kappa \,\middle|\, \alpha \in \operatorname{hspec}_{\left\{\frac{(K\cap G_n)N}{N}\right\}}(K/N)\right\} \subseteq \operatorname{hspec}_{\{K\cap G_n\}}(K)$$

dugu. Horrela, Lema 3.8 erabiliz, $[\eta,\kappa] \subseteq \operatorname{hspec}_{\{G_n\}}(G)$ dela ondorioztatzen dugu. □

Azpiatal hau bukatzeko Klopsch, Thillaisundaram eta Zugadi-Reizabalek [48]n frogatu zuten emaitza bat enuntziatuko dugu. Irizpide hau baliagarria izanen da $[0,1]$ espektrodun talde profinituak aurkitzeko.

Teorema 3.13 (cf. [48, Teorema 5.4])**.** *Izan bitez G oinarri kontagarriko pro-p taldea eta $\{G_n\}_{n\in\mathbb{N}}$ filtrazio serie normala. Demagun finituki sortutako $H \leq_c G$ azpitalde itxi guztiek $\operatorname{hdim}_{\{G_n\}}(H) = 0$ betetzen dutela. Orduan, $\operatorname{hspec}_{\{G_n\}}(G) = [0,1]$ da.*

3.1.3 Hausdorffen dimentsio R-estandarra

Talde R-analitiko trinkoetan filtrazio serie natural bat dugu eskuragarri. Izan bitez G d dimentsioko talde R-analitiko trinkoa eta (S,ϕ) N mailako azpitalde ireki R-estandarra. Gogoratu (1.6)n aurkeztu dugun filtrazio serie R-estandarra:

$$S_n := \phi^{-1}\left(\left(\mathfrak{m}^{N+n}\right)^{(d)}\right), \ \forall n \in \mathbb{N}_0.$$

Horiek azpitalde irekiak dira, eta Krullen Ebakidura Teorema (loc. cit.) dela eta, $\{S_n\}_{n\in\mathbb{N}}$ bada filtrazio serie bat. Orobat, (1.5)en ondorioz, $S_n \trianglelefteq S$ da $n \in \mathbb{N}$ guztietarako (ohartu S_n azpitaldeak S-n direla normalak, baina ez dute zertan G-n normalak izan), eta, beraz, (3.2) formulak S talde R-estandarrean $\{S_n\}_{n\in\mathbb{N}}$ seriearekin balio du.

Hausdorffen dimentsioa filtrazio seriearen menpekoa denez, hasiera batean bederen, talde R-analitiko trinko batean Hausdorffen dimentsioa, edota kutxa-dimentsioa, filtrazio R-estandarraren menpekoak izan zitezkeen. Haatik, benetan ez da hala: ikusiko denez, dimentsioak filtrazio serie R-estandarrekiko independeak dira. Hori frogatu aitzinetik, ordea, (1.7)ren ondorio hau aztertuko dugu:

Oharra 3.14. Izan bedi $(R, \mathfrak{m})$ pro-p domeinua. Orduan, $(R, \mathfrak{m})$-ren *Hilberten funtzioa* $H\colon \mathbb{N}_0 \to \mathbb{N}$, $n \mapsto \dim_{R/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1})$ da, eta n behar bezain handirako $p(n)$ funtzio polinomiko baten berdina da. Polinomio horrek $\dim_{\mathrm{Krull}}(R) - 1$ maila du eta R-ren *Hilberten polinomioa* deitzen zaio (ikusi [25, Kapitulua 6, Teorema C]). Horrenbestez, Euler-Maclaurinen formula dela eta, $\sum_{i=1}^{n-1} p(i)$ batura asintotikoki $\dim_{\mathrm{Krull}} R$ mailako $f(n)$ polinomio bati baliokidea da, hau da, haien zatidurak 1 baliora konbergitzen du n infinitura doan heinean.

Izan bitez, orain, q $R/\mathfrak{m}$ hondar gorputzaren kardinalitatea eta (S, ϕ) d dimentsioko eta N mailako talde R-estandarra. Orduan, (1.7)ren ondorioz,

$$\log_q |S : S_n| = d \sum_{i=N}^{N+n-1} H(i)$$

balioa $df(n)$-ri asintotikoki baliokidea da.

Emaitza honetan, behe kutxa-dimentsioa filtrazio estandarrarekiko independentea dela ikusiko dugu:

Lema 3.15 (cf. [27, Teorema 3.1])**.** *Izan bedi G talde R-analitiko trinkoa eta izan bitez (S, ϕ) eta (T, ψ) bi azpitalde ireki R-estandar. Orduan, $X \subseteq G$ guztietarako*

$$\mathrm{lbdim}_{\{S_n\}}(X) = \mathrm{lbdim}_{\{T_n\}}(X)$$

da.

Oharra. Goi kutxa dimentsiorako berdina gertatzen da, froga berarekin.

Froga. Izen bitez $N(S)$ eta $N(T)$, hurrenez hurren, S eta T azpitaldeen mailak. Lehenik eta behin, frogatu behar dugu badaudela bi zenbaki oso, $a, b \in \mathbb{N}$, non $n - b \in \mathbb{N}$ betetzen duten $n \in \mathbb{N}$ zenbaki oso guztietarako

$$S_{n+a} \leq T_n \leq S_{n-b} \tag{3.3}$$

den. Azken batean, $\psi \circ \phi^{-1}$ funtzio R-analitikoa $\phi(S \cap T) \subseteq \left(\mathfrak{m}^{N(S)}\right)^{(d)}$-n konbergentea denez eta $\psi \circ \phi^{-1}(\mathbf{0}) = \mathbf{0}$ denez, (1.1)en arabera, existitzen da $L \geq N(S)$ non

$$\psi \circ \phi^{-1}\left(\left(\mathfrak{m}^{L+n}\right)^{(d)}\right) \subseteq (\mathfrak{m}^n)^{(d)}$$

den $n \in \mathbb{N}$ guztietarako. Hortaz, $a = L - N(S) + N(T)$ bada, $S_{n+a} \leq T_n$ da $n \in \mathbb{N}$ guztietarako. Era berean, $\phi \circ \psi^{-1}$ funtzioarekin argudio berak errepikatuz (3.3) dugu. Horrenbestez,

$$\begin{aligned}
\operatorname{lbdim}_{\{T_n\}}(X) &= \liminf_{n\to\infty} \frac{\log|XT_n : T_n|}{\log|G : T_n|} \\
&\leq \liminf_{n\to\infty} \frac{\log|XS_{n+a} : S_{n+a}|}{\log|G : S_{n+a}|} \cdot \frac{\log|G : S_{n+a}|}{\log|G : T_n|} \\
&= \liminf_{n\to\infty} \frac{\log|XS_{n+a} : S_{n+a}|}{\log|G : S_{n+a}|} \cdot \frac{\log|G : S_{n+a}|}{\log|G : S_{n+a}| - \log|T_n : S_{n+a}|} \\
&= \operatorname{lbdim}_{\{S_n\}}(X),
\end{aligned}$$

azkenaurreko ezberdintzan

$$\lim_{n\to\infty} \frac{\log|G : S_{n+a}|}{\log|G : S_{n+a}| - \log|T_n : S_{n+a}|} = 1$$

dela erabilita. Izan ere, aurreko oharraren arabera,

$$\log|T_n : S_{n+a}| \leq \log|S_{n-b} : S_{n+a}| = \sum_{N+n-b}^{N+n+a-1} H(i)$$

da. Hortaz, n behar bezain handia denean, eskuineko terminoa $\dim_{\mathrm{Krull}}(R) - 1$ mailako $a + b$ funtzio polinomikoren batura da, eta $\log|G : S_{n+a}| = \log|G : S| + \log|S : S_{n+a}|$, aldiz, $\dim_{\mathrm{Krull}}(R)$ mailako funtzio polinomiko bati asintotikoki baliokidea da. Horrela, S eta T-ren paperak trukatuz bukatzen da froga. □

Hori dela eta, *behe kutxa-dimentsio R-estandarra edo estandarra*, $\operatorname{lbdim}_{\mathrm{st}}$ izendatuko duguna, defini daiteke, kontuan hartu gabe zein filtrazio serie estandarrekin ari garen lanean. Aipagarria da, R INDa denean, Lema 3.15en analogoa Hausdorffen dimentsiorako ere egia dela.

Lema 3.16. *Izan bitez R ideal nagusietako pro-p domeinua, G talde R-analitiko trinkoa eta (S,ϕ) eta (T,ψ) G-ren bi azpitalde R-estandar ireki. Orduan, edozein $X \subseteq G$ azpimultzotarako*

$$\mathrm{hdim}_{\{S_n\}}(X) = \mathrm{hdim}_{\{T_n\}}(X)$$

da.

Froga. Izan bedi q $R/\mathfrak{m}$ hondar-gorputzaren kardinalitatea. Horrela, R INDa denez, $|S_n : S_{n+1}| = |T_n : T_{n+1}| = q^d$ da $n \in \mathbb{N}$ guztietarako. Halaber, (3.3)ren arabera, existitzen dira bi zenbaki oso, $a, b \in \mathbb{N}$, non $S_{n+a} \le T_n \le S_{n-b}$ den $n - b \in \mathbb{N}$ betetzen duten $n \in \mathbb{N}$ zenbaki oso guztietarako. Hortaz,

$$|G : T_n|^{-1} \ge |G : S_{n+a}|^{-1} = q^{-d(a+b)}|G : S_{n-b}|^{-1}$$

da. Beraz, δ_S eta δ_T hurrenez hurren, $\{S_n\}_{n\in\mathbb{N}}$ eta $\{T_n\}_{n\in\mathbb{N}}$ filtrazio serieek definitutako distantziak badira, orduan

$$\delta_T(x,y) \ge q^{-d(a+b)}\delta_S(x,y)$$

da. Era berean, S eta T trukatuz, froga dezakegu badagoela $C > 0$ konstantea non $C \cdot \delta_S(x,y) \ge \delta_T(x,y)$ den. Alegia, (G,δ_S) eta (G,δ_T) espazio metrikoen arteko identitate funtzioa bilipschitziarra da, eta emaitza Proposizioa 3.6ren ondorio zuzena da. □

Izan bitez G talde R-analitiko trinkoa eta $S \le_o G$ azpitalde ireki R-estandarra. Orduan, $\{S_n\}_{n\in\mathbb{N}}$ filtrazio serieak G-n zein S-n definitzen ditu Hausdorffen dimentsioak. Atalaren hasieran aurkeztutako notazioa mantenduz, bi dimentsio horiek $\mathrm{hdim}^G_{\{S_n\}}$ eta $\mathrm{hdim}^S_{\{S_n\}}$ moduan bereiziko ditugu.

Lema 3.17. *Izan bitez G talde R-analitiko trinkoa eta (S,ϕ) azpitalde R-estandar irekia.*

(i) *Izan bitez $H \le_c G$ eta $U \subseteq_o H$, orduan $\mathrm{hdim}^G_{\{S_n\}}(H) = \mathrm{hdim}^G_{\{S_n\}}(U)$ da.*

(ii) *Izan bedi $X \subseteq S$, orduan $\mathrm{hdim}^S_{\{S_n\}}(X) = \mathrm{hdim}^G_{\{S_n\}}(X)$ da.*

Bereziki, $\mathrm{hdim}^G_{\{S_n\}}(H) = \mathrm{hdim}^S_{\{S_n\}}(H \cap S)$ *da $H \le_c G$ azpitalde itxi guztietarako.*

Oharra 3.18. Aurreko emaitza *goi* kutxa-dimentsiorako ere betetzen da, zehatz-mehatz froga berarekin. Behe kutxa dimentsiorako, berriz, (ii) bakarrik betetzen da.

Froga. (i) Alde batetik, H talde trinkoa da, G talde trikoaren azpimultzo itxia baita, eta, horrenbestez, badago $g_i \in G$ elementu kopuru finitu bat non $H = \cup_{i=1}^{r} g_i U$ den. Hortaz, kontuan hartuta Hausdorffen dimentsioaren egonkortasun finitua eta ezker biderketa funtzioa isometria bat dela, Proposizioa 3.6 dela eta,

$$\mathrm{hdim}_{\{S_n\}}^{G}(H) = \mathrm{hdim}_{\{S_n\}}^{G}\left(\cup_{i=1}^{r} g_i U\right) = \max_{i=1,\ldots,r} \mathrm{hdim}_{\{S_n\}}^{G}(g_i U) = \mathrm{hdim}_{\{S_n\}}^{G}(U)$$

da.
(ii) Izan bitez δ_G eta δ_S, hurrenez hurren, $\{S_n\}_{n\in\mathbb{N}}$ filtrazio serieak G-n eta S-n definituriko metrikak. Orduan, $\delta_G(x,y) = |G:S|^{-1}\delta_S(x,y)$ da, eta, beraz, (S,δ_S)-tik (G,δ_G)-ra partekotasun funtzioa bilipschitzitzarra da. Horrela, formula Proposizioa 3.6ren ondorioa da.

Azkenik, $H \cap S \leq_o H$ azpitalde irekia denez, (i) eta (ii)tik ondorioztatu dezakegu

$$\mathrm{hdim}_{\{S_n\}}^{G}(H) = \mathrm{hdim}_{\{S_n\}}^{G}(H \cap S) = \mathrm{hdim}_{\{S_n\}}^{S}(H \cap S)$$

dela. □

Korolarioa 3.19. *Izan bitez G talde R-analitiko trinkoa eta (S,ϕ) eta (T,ψ) bi azpitalde R-estandar ireki. Orduan, edozein $H \leq_c G$ azpitalde itxitarako*

$$\mathrm{hdim}_{\{S_n\}}(H) = \mathrm{hdim}_{\{T_n\}}(H)$$

da.

Froga. Teorema 3.7, Lema 3.15 eta Oharra 3.18 –Lema 3.17(ii) behe kutxa dimentsiorako betetzen da– aintzat hartuta:

$$\begin{aligned}
\mathrm{hdim}_{\{S_n\}}^{G}(H) &= \mathrm{hdim}_{\{S_n\}}^{S}(H \cap S \cap T) = \mathrm{lbdim}_{\{S_n\}}^{S}(H \cap S \cap T) \\
&= \mathrm{lbdim}_{\{S_n\}}^{G}(H \cap S \cap T) = \mathrm{lbdim}_{\{T_n\}}^{G}(H \cap S \cap T) \\
&= \mathrm{lbdim}_{\{T_n\}}^{T}(H \cap S \cap T) = \mathrm{hdim}_{\{T_n\}}^{T}(H \cap S \cap T) \\
&= \mathrm{hdim}_{\{T_n\}}^{G}(H).
\end{aligned}$$

□

Horrenbestez, $H \leq_c G$ azpitalde itxi baten *Hausdorffen dimentsio estandarra* edo *R-estandarra*, $\mathrm{hdim}_{\mathrm{st}}(H)$ moduan adieraziko dena, $\mathrm{hdim}_{\{S_n\}}(H)$ gisa definitzen da $\{S_n\}_{n\in\mathbb{N}}$ *edozein* filtrazio R-estandar delarik. Hori kontuan hartuta, G-ren *Hausdorffen espektro estandarra* edo *R-estandarra*

$$\mathrm{hspec}_{\mathrm{st}}(G) := \{\mathrm{hdim}_{\mathrm{st}}(H) \mid H \leq_c G\}$$

da. Bestalde, Lema 3.17ren berehalako ondorio hau dugu:

Korolarioa 3.20. *Izan bitez G talde R-analitiko trinkoa eta (S, ϕ) azpitalde R-estandar irekia. Orduan,* $\mathrm{hspec}_{\mathrm{st}}(G) = \mathrm{hspec}_{\mathrm{st}}(S)$ *da.*

Hori dela eta, talde R-analitiko trinko baten Hausdorffen espektro estandarra aztertzerakoan jatorrizko taldea bera R-estandarra dela asumi dezakegu.

Azpiatal hau bukatzeko azter dezagun azpitalde analitikoen eta zatidura talde analitikoen Hausdorffen dimentsio estandarra. Konkretuago, honako lema honek $\mathrm{hdim}^H_{\mathrm{st}}$ dimentsioa eta $\{S_n\}_{n\in\mathbb{N}}$ filtrazio serie R-estandarrak H-n induzitzen duen Hausdorffen dimentsioa, $\mathrm{hdim}^H_{\{H\cap S_n\}}$, erlazionatzen ditu.

Lema 3.21. *Izan bitez G talde R-analitiko trinkoa eta H azpitalde R-analitikoa. Orduan,* $\mathrm{hdim}^H_{\{H\cap S_n\}}(K) = \mathrm{hdim}^H_{\mathrm{st}}(K)$ *da $K \leq_c H$ azpitalde itxi guztietarako eta edozein G-ren $\{S_n\}_{n\in\mathbb{N}}$ filtrazio serie R-estandarrerako.*

Froga. Lehenik eta behin, izan bitez G taldearen bi filtrazio serie R-estandar, $\{S_n\}_{n\in\mathbb{N}}$ eta $\{T_n\}_{n\in\mathbb{N}}$. Lema 3.8 eta Korolarioa 3.19 direla eta, berehalakoa da

$$\mathrm{hdim}^H_{\{H\cap S_n\}}(K) = \mathrm{hdim}^H_{\{H\cap T_n\}}(K),\ \forall K \leq_c H \tag{3.4}$$

dela ikustea.

Bigarrenik, ikus dezagun existitzen dela $S \leq_o G$ azpitalde R-estandar irekia non $\{H \cap S_n\}_{n\in\mathbb{N}}$ seriea H-ren filtrazio serie R-estandarra den. Behin hori frogatuta, (3.4)ren eraginez, edozein $\{T_n\}_{n\in\mathbb{N}}$ filtrazio serie R-estandarretarako eta edozein $K \leq_c H$ azpitalde itxitarako

$$\mathrm{hdim}^H_{\{H\cap T_n\}}(K) = \mathrm{hdim}^H_{\{H\cap S_n\}}(K) = \mathrm{hdim}^H_{\mathrm{st}}(K)$$

izanen genuke, nahi dugun moduan.

Izan bitez $d = \dim G$ eta $k = \dim H$. Orduan, H azpitalde R-analitikoa denez, Definizioa 1.48ren arabera, identitatearen (U, ϕ) R-karta existitzen da non $\phi(1) = \mathbf{0}$ eta

$$\begin{aligned}\phi(H \cap U) &= \left\{(x_1, \ldots, x_d) \in \left(\mathfrak{m}^L\right)^{(d)} \,\middle|\, x_{k+1} = \cdots = x_d = 0\right\} \\ &= \left(\mathfrak{m}^L\right)^{(k)} \times \{0\}^{(d-k)}\end{aligned}$$

diren, $N \geq 1$ egoki batentzat. Are gehiago, Lema 1.21 dela eta, U irekiaren barruan $S \leq_o G$ azpitalde ireki R-estandar bat dago, $L \geq N$ mailakoa, eta dagokion homeomorfismoa $\phi|_S$ da. Beraz,

$$\begin{aligned}\phi\left(H\cap S\right) &= \phi\left(S\right)\cap\phi(H\cap U)\\ &= \left(\mathfrak{m}^N\right)^{(d)}\cap\left(\left(\mathfrak{m}^N\right)^{(k)}\times\{0\}^{(d-k)}\right) = \left(\mathfrak{m}^N\right)^{(k)}\times\{0\}^{(d-k)}.\end{aligned}$$

Hortaz, $\mathrm{pr}\colon (\mathfrak{m}^L)^{(k)}\times\{0\}^{(d-k)}\to(\mathfrak{m}^L)^{(k)}$ proiekzio naturala bada, $(H\cap S,\psi)$, non $\psi=\mathrm{pr}\circ\phi|_{H\cap S}$, H-ren azpitalde R-estandar irekia da. Gainera,

$$\psi\left(H\cap S_n\right)=\mathrm{pr}\left(\phi(H\cap U)\cap\phi\left(S_n\right)\right)=\left(\mathfrak{m}^{L+n}\right)^{(k)}$$

da, eta, ondorioz, $\{H\cap S_n\}_{n\in\mathbb{N}}$ H-ren filtrazio serie R-estandarra da. □

Zatiduretarako, gogoratu G talde R-analitiko trinkoa eta $N\trianglelefteq G$ azpitalde analitiko normala badira, G/N ere talde R-analitiko trinkoa dela Proposizioa 1.59ren arabera. Horrenbestez, erlaziona ditzagun zatiduraren espektro R-estandarra eta $\{S_nN/N\}_{n\in\mathbb{N}}$ filtrazio seriearekiko lortzen dena.

Lema 3.22. *Izan bitez G talde R-analitiko trinkoa, $\{S_n\}_{n\in\mathbb{N}}$ filtrazio serie R-estandarra eta $N\trianglelefteq G$ azpitalde R-analitiko normala. Orduan,*

$$\mathrm{hdim}_{\mathrm{st}}(H)=\mathrm{hdim}_{\left\{\frac{S_nN}{N}\right\}}(H)$$

da $H\leq_c G/N$ guztietarako.

Froga. Hasteko finka dezagun notazioa: $d=\dim G$ eta $e=\dim G/N$ dira, $\pi\colon G\to G/N$ epimorfismo kanonikoa eta $\mathrm{pr}\colon\mathfrak{m}^{(d)}\to\mathfrak{m}^{(e)}$ azken e koordenatuetara proiekzioa.

Lehenik eta behin, $\{S_n\}_{n\in\mathbb{N}}$ eta $\{T_n\}_{n\in\mathbb{N}}$ bi filtrazio serie R-estandar badira, Lema 3.15en froga hitzez hitz errepikatuz eta Teorema 3.7 dela eta, ikus daiteke $H\leq_c G/N$ guztietarako

$$\mathrm{hdim}_{\left\{\frac{S_nN}{N}\right\}}(H)=\mathrm{lbdim}_{\left\{\frac{S_nN}{N}\right\}}(H)=\mathrm{lbdim}_{\left\{\frac{T_nN}{N}\right\}}(H)=\mathrm{hdim}_{\left\{\frac{T_nN}{N}\right\}}(H)$$

dela. Ondorioz, nahikoa da $S\leq_o G$ azpitalde R-estandar ireki bat topatzea non $\{S_nN/N\}_{n\in\mathbb{N}}$ G/N-ren filtrazio serie R-estandarra den. Lema 1.58ren arabera, badago N-ra moldatutako identitatearen (U,ϕ) R-karta, hau da, $\phi(1)=\mathbf{0}$ da eta $\mathrm{pr}\circ\phi(x)=\mathrm{pr}\circ\phi(y)$ da baldin eta soilik baldin $xy^{-1}\in N$ bada. Bestetik, Lema 1.21en arabera, U-ren barruan badago $S\leq G$ azpitalde R-estandar irekia, L mailakoa eta $\phi|_S$ homeomorfismoarekin. Izan bedi $\sigma\colon\pi(S)\to S$ sekzio jarraitu bat eta demagun $\sigma(1N) = 1$ dela, [64, Proposizioa 2.2.2] dela eta badago halakorik. Orduan, $\pi(S)$ G/N-ren talde R-estandarra da, L mailakoa, e dimentsiokoa eta $\psi = \mathrm{pr} \circ \phi \circ \sigma$ homeomorfismoarekin. Ohartu (U, ϕ) karta N-ra moldatua egoteagatik, ψ-ren definizioa hautatutako sekzioarekiko independentea dela eta $\psi(S_n N/N) = \mathrm{pr} \circ \phi(S_n) = (\mathfrak{m}^{L+n})^{(e)}$ dela. Alegia, $\{S_n N/N\}_{n\in\mathbb{N}}$ G/N-ren filtrazio serie R-estandarra da. □

3.2 Hausdorffen dimentsioa azpibarietateetan

2017. urteko [27] artikuluan autoreek azpitalde R-analitikoen dimentsio analitikoaren eta Hausdorffen dimentsioaren arteko erlazio aztertzen dute. Oro har azterketa guztia hurrengo kasu partikularrera murrizten da.

Lema 3.23. *Izan bedi $(R, \mathfrak{m})$ pro-p domeinua. Orduan, $\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)}$ azpitaldearen kutxa-dimentsioa $\mathfrak{m}^{(d)}$ talde batukorrean eta $\left\{(\mathfrak{m}^n)^{(d)}\right\}_{n\in\mathbb{N}}$ filtrazio seriearekiko k/d balio duen limite propioa da.*

Oharra. Era berean, $\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)}$ multzoaren Hausdorffen dimentsioa ere k/d da, azpitalde itxia baita (ikusi Teorema 3.7).

Froga. Izan bedi q $R/\mathfrak{m}$ hondar gorputzaren kardinalitatea, eta oroitu Oharra 3.14ren arabera,

$$\left|\mathfrak{m}^{(d)} : (\mathfrak{m}^n)^{(d)}\right| = q^{d\sum_{i=1}^{n-1} H(i)}$$

dela H R-ren Hilberten funtzioa izanik, horrela $n \geq N$ denean,

$$\begin{aligned}&\left|\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)} : \left(\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)}\right) \cap (\mathfrak{m}^n)^{(d)}\right| \\ &\qquad = \left|\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)} : (\mathfrak{m}^n)^{(k)} \times \{0\}^{(d-k)}\right| = q^{k\sum_{i=N}^{n-1} H(i)}\end{aligned}$$

da. Hori dela eta,

$$\begin{aligned}&\mathrm{lbdim_{st}}\left(\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)}\right) \\ &\qquad = \liminf_{n\to\infty} \frac{\log_q \left|\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)} : \left(\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)}\right) \cap (\mathfrak{m}^n)^{(d)}\right|}{\log_q \left|\mathfrak{m}^{(d)} : (\mathfrak{m}^n)^{(d)}\right|} \\ &\qquad = \liminf_{n\to\infty} \frac{k\sum_{i=N}^{n-1} H(i)}{d\sum_{i=1}^{n-1} H(i)} = \frac{k}{d}.\end{aligned}$$

Kontuan hartuta azken berdintzan

$$\lim_{n\to\infty} \frac{H(1)+\cdots+H(n-1)}{H(N)+\cdots+H(n-1)} = 1 + \lim_{n\to\infty} \frac{H(1)+\cdots+H(N-1)}{H(N)+\cdots+H(n-1)} = 1$$

dela. Izan ere, erdiko zatikian izendatzailea asintotikoki funtzi polinomiko ez-konstante baten baliokidea da, eta zenbakitzailea, berriz, konstantea da. Bereziki, dimentsioa limite propio bat da. □

Datozen korolarioetan (3.1) identitatea frogatuko dugu.

Korolarioa 3.24. *Izan bitez G talde R-analitiko trinkoa eta $M \subseteq G$ azpibarietate itxia. Orduan,*

$$\operatorname{ubdim}_{\text{st}} M = \frac{\max\{\dim_x M \mid x \in M\}}{\dim G}$$

da, eta dimentsio hori limite propioa da.

Oharra 3.25. Bereziki, limitea propioa denez, M azpibarietatearen *kutxa-dimentsio estandarraz*, $\operatorname{bdim}_{\text{st}} M$, hitz egin dezakegu inolako anbiguotasunik gabe.

Froga. Izan bedi d talde analitikoaren dimentsioa. Definizioa 1.48ren arabera, $x \in M$ bakoitzerako badago $U_x \subseteq_o M$ irekia zeinak x barruan duen eta x-ren (V_x, ψ) R-karta erregularra non $U_x \subseteq V_x$ eta

$$\psi(U_x) = \{(x_1, \ldots, x_d) \in \psi(V_x) \mid x_{k+1} = \cdots = x_d = 0\},$$

$k = \dim_x M$, diren. Bestetik, behar izanez gero azpimultzo batera pasatuz, asumi dezagun orokortasunik galdu gabe $V_x = xS$ dela, $S \leq_o G$ azpitalde R-estandar egoki batentzat $\phi\colon S \to \left(\mathfrak{m}^N\right)^{(d)}$, $y \mapsto \psi(y)$ karta globalarekin. Alegia,

$$\phi\left(x^{-1}U_x\right) = \psi(U_x) = \left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)}. \tag{3.5}$$

Bestalde, Lema 1.20 berridazteko modu bat da ϕ isometria bat dela esatea, S-n $\{S_n\}_{n\in\mathbb{N}}$ filtrazioak definitzen duen espazio metrikoaren eta $\left(\mathfrak{m}^N\right)^{(d)}$ talde batukorraren artean $\left\{\left(\mathfrak{m}^{N+n}\right)^{(d)}\right\}_{n\in\mathbb{N}}$ filtrazio serieak definitzen duen metrikarekin. Hori dela eta, ezker biderketa funtzioa isometria bat denez, Oharra 3.18 dela eta,

$$\operatorname{ubdim}^G_{\{S_n\}}(U_x) = \operatorname{ubdim}^G_{\{S_n\}}(x^{-1}U_x) = \operatorname{ubdim}^S_{\{S_n\}}(x^{-1}U_x)$$

da. Areago, hurrenez hurren, Proposizioa 3.6, (3.5) identitatea eta Lema 3.23 erabilita

$$\mathrm{ubdim}^{G}_{\{S_n\}}(U_x) = \mathrm{ubdim}^{S}_{\{S_n\}}(x^{-1}U_x) = \mathrm{ubdim}^{(\mathfrak{m}^N)^{(d)}}_{\{(\mathfrak{m}^{N+n})^{(d)}\}}(\phi(x^{-1}U_x)) = \frac{\dim_x M}{\dim G} \tag{3.6}$$

da eta dimentsio hori limite propioa da. Alde batetik, monotoniagatik

$$\frac{\dim_x M}{\dim G} = \mathrm{ubdim}_{\mathrm{st}}(U_x) \leq \mathrm{ubdim}_{\mathrm{st}}(M) \tag{3.7}$$

da $x \in M$ guztietarako. Beste alde batetik, M trinkoa denez –G talde trinkoaren azpimultzo itxia da–, badaude $x_i \in M$ elementuak, kopuru finitu bat, non $M = \cup_{i\in I} U_{x_i}$ den. Horrenbestez, egonkortasun finitua eta (3.6) kontuan hartuta,

$$\mathrm{ubdim}_{\mathrm{st}}(M) = \mathrm{ubdim}_{\mathrm{st}}\left(\cup_{i\in I} U_{x_i}\right) = \max_{i\in I} \mathrm{ubdim}_{\mathrm{st}}(U_{x_i}) \leq \frac{\max\{\dim_x M \mid x \in M\}}{\dim G} \tag{3.8}$$

da. Horrek eta (3.7) identitateak ematen dute emaitza. □

Hausdorffen dimentsiorako pareko emaitza frogatzeko, aurreko argudioak apur bat findu behar dira.

Korolarioa 3.26. *Izan bitez G talde R-analitiko trinkoa, $\{S_n\}_{n\in\mathbb{N}}$ filtrazio serie R-estandarra eta M azpibarietate itxia. Orduan,*

$$\mathrm{hdim}_{\{S_n\}}(M) = \frac{\max\{\dim_x M \mid x \in M\}}{\dim G}$$

da.

Froga. Frogak aurreko frogaren traza du, eta hango notazioa mantenduko dugu. Besteak beste, $x \in M$ bakoitzerako badaude $U_x \subseteq_o M$ azpimultzo irekia eta (S, ϕ) azpitalde R-estandar irekia non $x^{-1}U_x \subseteq S$ eta

$$\phi(x^{-1}U_x) = \left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)},$$

$k = \dim_x M$, diren. Demagun δ_S dela S azpitalde R-estandarrak G-n definitutako distantzia, orduan (3.6) berdintza frogatzeko erabiliriko argudioak hitzez hitz errepikatuz

$$\mathrm{hdim}_{\delta_S}(U_x) = \frac{\dim_x M}{\dim G}$$

dela froga daiteke. Izan bitez orain (T,ψ) beste azpitalde R-estandar ireki bat eta δ_T azpitalde hori erabilita G-n definitzen den distantzia. Alde batetik, [26, Proposizioa 3.4] eta (3.6) kontuan hartuta,

$$\operatorname{hdim}_{\delta_T}(U_x) \leq \operatorname{lbdim}_{\mathrm{st}}(U_x) \leq \operatorname{ubdim}_{\mathrm{st}}(U_x) = \frac{\dim_x M}{\dim G}$$

da. Definitu $\tilde{U}_x = x^{-1}U_x \cap T$, orduan $\psi(\tilde{U}_x) \subseteq \psi(S \cap T)$ da, eta, beraz,

$$\phi(\tilde{U}_x) = \phi(x^{-1}U_x \cap T) = \left(\left(\mathfrak{m}^N\right)^{(k)} \times \{0\}^{(d-k)}\right) \cap \phi(S \cap T),$$

denez eta $\phi(S \cap T)$ irekia denez $\mathfrak{m}^{(d)}$-n, badago $K \in \mathbb{N}$ non

$$\left(\mathfrak{m}^K\right)^{(k)} \times \{0\}^{(d-k)} \subseteq \phi(\tilde{U}_x)$$

den. Deitu δ $\mathfrak{m}^{(d)}$-n $\left\{\left(\mathfrak{m}^N\right)^{(d)}\right\}_{N\in\mathbb{N}}$ filtrazio seriea erabilita definitzen den distantziari. Horrela, ϕ eta ψ isometriak direnez, $\phi \circ \psi^{-1}$ funtzioa ere isometria bat da, $(\psi(S \cap T), \delta)$-tik $(\phi(S \cap T), \delta)$-ra. Hortaz, Lema 3.17, Proposizioa 3.6 eta monotonia kontuan hartuta,

$$\begin{aligned}
\operatorname{hdim}^G_{\delta_T}(U_x) &\geq \operatorname{hdim}^T_{\delta_T}\left(\tilde{U}_x\right) = \operatorname{hdim}^{\mathfrak{m}^{(d)}}_{\delta}\left(\psi(\tilde{U}_x)\right) \\
&= \operatorname{hdim}^{\mathfrak{m}^{(d)}}_{\delta}\left(\phi \circ \psi^{-1} \circ \psi(\tilde{U}_x)\right) = \operatorname{hdim}^{\mathfrak{m}^{(d)}}_{\delta}(\phi(\tilde{U}_x)) \\
&\geq \operatorname{hdim}^{\mathfrak{m}^{(d)}}_{\left\{(\mathfrak{m}^n)^{(d)}\right\}}\left(\left(\mathfrak{m}^K\right)^{(k)} \times \{0\}^{(d-k)}\right) = k/d
\end{aligned}$$

da, azken berdintzan Lema 3.23 erabilita. Hori dela eta, $\operatorname{hdim}(U_x) = \dim_x M / \dim G$ da, edozein delarik ere hasierako filtrazio serie estandarra. Bukatzeko nahikoa da (3.7) eta (3.8)-ko argudiak errepikatzea, baina kutxa-dimentsioaren ordez Hausdorffen dimentsioa hartuta. □

Azpitalde R-analitikoak azpibarietate R-analitiko puru itxiak direnez, [27]ko emaitza nagusia berreskura dezakegu:

Korolarioa 3.27 (cf. [27, Teorema Nagusia])**.** *Izan bitez G talde R-analitiko trinkoa eta H azpitalde R-analitikoa. Orduan,*

$$\operatorname{bdim}_{\mathrm{st}}(H) = \operatorname{hdim}_{\mathrm{st}}(H) = \frac{\dim H}{\dim G}$$

da. Bereziki, bi dimentsio horiek limite propioak dira.

Azkenik $R = \mathbb{Z}_p$ denean, azpitalde itxi guztiak talde p-adiko analitikoak dira (ikusi [24, Teorema 9.6]), eta, horrenbestez, Teorema 3.2ren beste froga bat eman dugu atal honetan. Are gehiago, bide batez, talde horietan Hausdorffen espektro estandarraren deskribapen argia dugu. Izan ere, G talde p-adiko analitiko d-dimentsional trinkoa bada, orduan

$$\mathrm{hspec}_{\mathrm{st}}(G) \subseteq \left\{0, \frac{1}{d}, \dots, \frac{d-1}{d}, 1\right\}.$$

3.3 Talde R-analitiko trinko abeldarrak

Jarrai dezagun talde R-analitiko profinituen espektro estandarrak deskribatzen. Noski, aurrekoa kontuan hartuta p-adiko analitikoak ez diren taldeetan jarriko dugu arreta, hots, R ez da $\mathbb{Z}_p$-ren finituki sortutako eraztun hedadura izanen. Lehenik eta behin, kasu abeldarra aztertuko dugu.

Oharra. Hemendik aurrera soilik azpitalde itxien Hausdorffen dimentsioaz arduratuko garenez, Teorema 3.7 inplizituki behin eta berriro erabiliko da.

Proposizioa 3.28. *Izan bitez R pro-p domeinua eta (S, ϕ) talde R-estandar abeldarra. Demagun R-k karakteristika positiboa duela edo R-ren Krullen dimentsioa gutxienez 2 dela. Orduan,* $\mathrm{hspec}_{\mathrm{st}}(S) = [0, 1]$ *da.*

Froga. Teorema 3.13ren arabera, nahikoa da finituki sortutako $H \leq_c S$ azpitalde guztietarako $\mathrm{hdim}_{\mathrm{st}}(H) = 0$ dela frogatzea. Izan bitez $d = \dim S$ eta $H \leq_c S$ topologikoki r elementuk sorturiko azpitalde itxia.

Demagun lehendabizi R-k p karakteristika positiboa duela, S-ko eragiketa talde eragiketa formal batek ematen duenez, (1.3)ren arabera, $x \in S_n$ bada

$$\phi(x^p) \equiv p\phi(x) = \mathbf{0} \mod \left(\mathfrak{m}^{2n}\right)^{(d)},$$

da, eta, beraz, $x^p \equiv 1 \pmod{S_{2n}}$ da. Hots, S_n/S_{2n} p-talde abeldar elementala da.

Bestalde, S abeldarra denez, $H/(H \cap S_n)$ taldea p-talde abeldarra da eta p^e exponentea du, $e \leq \lceil \log_2(n) \rceil$ batentzat. Halaber, H topologikoki r elementuk sortzen dutenez, $H/(H \cap S_n)$ taldea r elementuk sortzen dute. Hortaz,

$$|H : H \cap S_n| \leq p^{er} \leq p^{\lceil \log_2(n) \rceil r}.$$

Oharra 3.14ren arabera, $|R/\mathfrak{m}| = q = p^c$ bada, $\log_q |S : S_n|$ asintotikoki $df(n)$ funtzio polinomiko bati baliokidea da eta $f(n)$-ren polinomio maila $\dim_{\mathrm{Krull}}(R)$

da; horrenbestez,

$$\mathrm{hdim}_{\mathrm{st}}(H) = \liminf_{n\to\infty} \frac{\log_p |H : H \cap S_n|}{\log_p |S : S_n|} \leq \liminf_{n\to\infty} \frac{r\lceil \log_2(n)\rceil}{cdf(n)} = 0.$$

Era berean, R-k gutxienez 2 Krullen dimentsioa badu, (1.3) dela eta, $x \in S_n$ denean

$$\phi(x^p) \equiv p\phi(x) = \mathbf{0} \mod \left(\mathfrak{m}^{n+1}\right)^{(d)}$$

da, eta, beraz, S_n/S_{n+1} p-talde abeldar elementala da. Hortaz, $H/(H\cap S_n)$ taldea r elementuk sorturiko talde abeldarra da eta haren exponentea p^e bada, orduan $e \leq n-1$ da. Hori dela eta,

$$|H : H \cap S_n| = |H/(H \cap S_n)| \leq p^{re} \leq p^{r(n-1)},$$

eta, Oharra 3.14ren arabera,

$$\mathrm{hdim}_{\mathrm{st}}(H) = \liminf_{n\to\infty} \frac{\log_p |H : H \cap S_n|}{\log_p |S : S_n|} \leq \liminf_{n\to\infty} \frac{r(n-1)}{cdf(n)} = 0$$

da, $f(n)$-ren polinomio maila $\dim_{\mathrm{Krull}}(R) \geq 2$ baita. □

Korolarioa 3.20ren eraginez, aurreko emaitza talde R-analitiko trinkoetara orokortu daiteke.

Korolarioa 3.29. *Izan bitez R pro-p domeinua eta G talde R-analitiko trinko abeldarra. Demagun R-k karakteristika positiboa duela edo R-ren Krullen dimentsioa gutxienez 2 dela. Orduan,* $\mathrm{hspec}_{\mathrm{st}}(G) = [0,1]$ *da.*

Orobat, jakina da bat dimentsio analitikoko talde R-estandarrak abeldarrak direla (ikusi [35, Teorema 1.6.7]), eta, ondorioz, honakoa dugu:

Korolarioa 3.30. *Izan bedi G bat dimentsioko talde R-analitiko trinkoa. Demagun R pro-p domeinuak karakteristika positiboa edo gutxienez 2 Krullen dimentsioa duela. Orduan,* $\mathrm{hspec}_{\mathrm{st}}(G) = [0,1]$ *da.*

3.4 Talde $\mathbb{F}_p[[t]]$-analitiko trinkoak

Atala 3.3k hau iradokitzen du: G talde R-analitiko (ez p-adiko analitiko) trinkoa ebazgarria denean, haren espektro estandarra $[0,1]$ tarte erreal osoa da. Hori frogatzeko estrategia espektroan elkarren segidako tarteak txertatzean datza, serie ebazgarriaren ondoz ondoko zatidurak abeldarrak direla erabilita. Izan ere, emaitza hau dugu:

Lema 3.31. *Izan bitez G talde R-analitiko trinkoa eta $N \trianglelefteq K \leq G$ talde R-analitikoak. Demagun* $\mathrm{hspec}_{\mathrm{st}}(K/N) = [0,1]$ *dela. Orduan,*

$$\left[\frac{\dim N}{\dim G}, \frac{\dim K}{\dim G}\right] = [\mathrm{hdim}_{\mathrm{st}}(N), \mathrm{hdim}_{\mathrm{st}}(K)] \subseteq \mathrm{hspec}_{\mathrm{st}}(G)$$

da.

Froga. Lehenengo berdintza Korolarioa 3.27ren ondorioa da, izan ere $\mathrm{hdim}_{\mathrm{st}}(H) = {}^{\dim H}/_{\dim G}$ da eta dimentsio hori limite propioa da. Bestetik, inklusioa berehalakoa da Korolarioa 3.12, Lema 3.21 eta Lema 3.22 direla eta. □

Horrenbestez, talde R-analitiko trinkoetan azpitalde R-analitikoak bilatzeko irizpide bat behar dugu. Horretarako oztopo nagusia, Lieren teoria klasikoarekin alderatuta, hau da: jakina da Lieren talde erreal (edo p-adiko) batean azpitalde itxi guztiak Lieren azpitaldeak direla. Alabaina, talde R-analitikoetan itxia izatea beharrezko baldintza da (ikusi Lema 1.57), baina ez da nahikoa. Adibidez, hartu $\mathbb{F}_p[[t]]$ talde $\mathbb{F}_p[[t]]$-estandarra, $\mathbb{F}_p[\![t^2]\!]$ azpitalde itxia bere kabuz talde $\mathbb{F}_p[[t]]$-analikoa da, baina barietate egitura horiek ez dira bateragarriak eta $\mathbb{F}_p[\![t^2]\!]$ ez da azpitalde $\mathbb{F}_p[[t]]$-analitikoa.

Hemendik aurrera $R = \mathbb{F}_p[[t]]$ hartuko dugu. Kasu horretan, azpitalde $\mathbb{F}_p[[t]]$-analitikoak Proposizioa 1.53ren bidez topa daitezke. Emaitza horren arabera, azpimultzo analitikoek azpibarietate $\mathbb{F}_p[[t]]$-analitiko egitura dute. Oroitu $X \subseteq M$ azpimultzo analitikoa dela, $x \in X$ bakoitzerako existitzen badira x-ren U ingurune irekia eta U-n definituriko zenbait funtzio $\mathbb{F}_p[[t]]$-analitiko, $f_1, \ldots, f_r$ ($r = r_x$ baterako), non

$$X \cap U = \{y \in U \mid f_i(y) = 0 \ \forall i = 1, \ldots, r\}$$

den.

Teorema 3.32 (cf. [45, Korolarioa 4.2])**.** *Izan bedi G talde $\mathbb{F}_p[[t]]$-analitikoa. Demagun $H \leq G$ azpitaldea eta azpimultzo analitikoa dela. Orduan, H azpitalde $\mathbb{F}_p[[t]]$-analitikoa da.*

Ikus ditzagun aitzineko teoremaren pare bat aplikazio:

Korolarioa 3.33. *Izan bitez S talde $\mathbb{F}_p[[t]]$-estandarra eta $a \in S$. Orduan, $Z(S)$ eta $C_S(a)$ azpitalde $\mathbb{F}_p[[t]]$-analitikoak dira.*

Froga. Aurreko teoremaren arabera, nahikoa da $Z(S)$ eta $C_S(a)$ multzo analitikoak direla frogatzea. Lehenengoa, hain zuzen ere [45, Korolarioa 4.3]n frogatzen da, eta bigarrena era berean egiten da. Izan ere, S talde $\mathbb{F}_p[[t]]$-estandarra denez, $\left(t^N\right)^{(d)}$-rekin identifikatu daiteke (N maila eta d dimentsioa dira), eta biderketa talde eragiketa formal batek definitzen du. Beraz, (1.5) dela eta, badaude $g_{i,\alpha} \in \mathbb{F}_p[[t]][[X_1, \ldots, X_d]]$ berretura serieak non

$$\pi_i\left(y^{-1}ay\right) = a_i + \sum_{|\alpha|\geq 1} g_{i,\alpha}(a) y_1^{\alpha_1} \ldots y_d^{\alpha_d} = a_i + h_i(y)$$

den $y \in S$ guztietarako –hemen $\pi_i\colon \left(t^N\right)^{(d)} \to \left(t^N\right)$ funtzioa $i$garren koordenatura proiekzioa da–. Bestalde, $h_i(y) = \sum_{|\alpha|\geq 1} g_{i,\alpha}(a) y_1^{\alpha_1} \ldots y_d^{\alpha_d}$ funtzioak modu argian $\mathbb{F}_p[[t]]$-analitikoak dira, eta, horrenbestez,

$$\begin{aligned} C_S(a) &= \left\{y \in S \mid \pi_i\left(y^{-1}ay\right) = a_i \ \forall i = 1, \ldots, d\right\} \\ &= \{y \in S \mid h_i(y) = 0 \ \forall i = 1, \ldots, d\} \end{aligned}$$

da. Bereziki, $C_S(a)$ multzo analitikoa da. □

Bigarren aplikazioak $\mathrm{GL}_n(K)$ talde lineal okorrarekin du zerikusia. Talde horretan ohiko topologiaz gain, R-ren topologia $\mathfrak{m}$-adikotik eratorritakoa, Zariskiren topologia dago. Bigarren horretan azpimultzo itxiak *multzo afinak* dira, hau da, forma honetako azpimultzoak:

$$\{A \in \mathrm{GL}_n(R) \mid f(A) = 0, \ \forall f \in \mathcal{F}\},$$

non $\mathcal{F} \subseteq R[\mathbf{X}]$ polinomio familia bat den ($\mathbf{X}$-n n^2 aldagai daude). Era berean, zein-nahi $H \leq \mathrm{GL}_n(R)$ azpitalde ohiko topologiaren azpiespazio topologiarekin eta Zariskiren topologiarekin horni daiteke, bigarrena ahulagoa izanik (funtzio polinomikoak jarraituak dira topologia $\mathfrak{m}$-adikoarekiko).

Zariskiren topologiari dagokionez, hurrengoak beharko ditugu:

Proposizioa 3.34 (cf. [73, Lema 5.9 eta Teorema 5.11])**.** *Izan bitez $H \leq \mathrm{GL}_n(R)$ eta $\mathcal{H}$ haren Zariski itxitura.*

(i) *Orduan, $\mathcal{H} \leq \mathrm{GL}_n(R)$ da.*

(ii) *Demagun $H \trianglelefteq \mathrm{GL}_n(R)$ dela, orduan $\mathcal{H} \trianglelefteq \mathrm{GL}_n(R)$ da.*

(iii) *Demagun H nilpotentea dela eta c nilpotentzia klasea duela. Orduan, $\mathcal{H}$-k c luzerako serie zentral bat du non serieko azpitalde guztiak Zariski itxiak diren. Bereziki, $\mathcal{H}$ nilpotentea da eta c nilpotentzia klasea du.*

(iv *Demagun H ebazgarria dela eta c ebazgarritasun luzera duela. Orduan, $\mathcal{H}$-k ℓ luzerako serie subnomal bat du non serieko azpitalde guztiak Zariski itxiak diren. Bereziki, $\mathcal{H}$ ebazgarria da eta ℓ ebazgarritasun luzera du.*

(iv) *Izan bedi $K \leq H$ azpitaldea eta demagun $|H : K|$ finitua dela. Izan bedi $\mathcal{K}$ K azpitaldearen Zariski itxitura $\mathrm{GL}_n(K)$-n, orduan $|\mathcal{H} : \mathcal{K}|$ finitua da.*

Korolarioa 3.35. *Izan bitez $G \subseteq \mathrm{GL}_n(\mathbb{F}_p[[t]])$ talde $\mathbb{F}_p[[t]]$-analitiko lineala eta $\mathcal{H} \leq \mathrm{GL}_n(\mathbb{F}_p[[t]])$ azpitalde Zariski itxia. Orduan, $\mathcal{H} \cap G$ azpitalde $\mathbb{F}_p[[t]]$-analitikoa da G-n.*

Froga. Teorema 3.32 kontuan hartuta $\mathcal{H} \cap G$ multzo $\mathbb{F}_p[[t]]$-analitikoa dela frogatu behar da. Alde batetik, $\mathcal{H}$ Zariski itxia denez, multzo afina da, hau da, existitzen da $\mathcal{F} \subseteq \mathbb{F}_p[[t]][\mathbf{X}]$ polinomio multzoa ($\mathbf{X}$ tuplak n^2 aldagai ditu) non

$$\mathcal{H} = \{A \in \mathrm{GL}_n(\mathbb{F}_p[[t]]) \mid f(A) = 0 \ \forall f \in \mathcal{F}\}$$

den. Areago, $\mathbb{F}_p[[t]][\mathbf{X}]$ noetherdarra denez, $\mathcal{F}$ finitua dela suposa dezakegu, eta, beraz,

$$\mathcal{H} \cap G = \{A \in G \mid f(A) = 0 \ \forall f \in \mathcal{F}\}$$

azpimultzo $\mathbb{F}_p[[t]]$-analitikoa da. □

Orain, aurreko emaitzak bilduz, Teorema 3.3(ii) froga dezakegu:

Teorema 3.36. *Izan bedi G talde $\mathbb{F}_p[[t]]$-analitiko trinko ebazgarria. Orduan,*

$$\mathrm{hspec}_{\mathrm{st}}(G) = [0, 1]$$

da.

Froga. Korolarioa 3.20 dela eta, orokortasunik galdu gabe G talde R-estandarra dela suposa dezakegu. Lehendabizi, G taldea $\mathbb{F}_p[[t]]$-ren gainean lineala denean frogatuko dugu baieztapena. Hots, demagun $G \subseteq \mathrm{GL}_n(\mathbb{F}_p[[t]])$ dela, eta izan bedi $\mathcal{G}$ G-ren Zariski itxitura $\mathrm{GL}_n(\mathbb{F}_p[[t]])$-n. Proposizioa 3.34ren arabera, $\mathcal{G}$ talde ebazgarria da eta badago

$$\mathcal{G} = \mathcal{H}_1 \trianglerighteq \mathcal{H}_2 \trianglerighteq \cdots \trianglerighteq \mathcal{H}_{\ell-1} \trianglerighteq \mathcal{H}_\ell = \{1\}$$

azpitalde Zariski itxiz osaturiko serie ebazgarria, hau da, ondoz ondoko zatidurak talde abeldarrak dira. Beraz, Korolarioa 3.35 dela eta,

$$G = \mathcal{H}_1 \cap G \trianglerighteq \mathcal{H}_2 \cap G \trianglerighteq \cdots \trianglerighteq \mathcal{H}_{\ell-1} \cap G \trianglerighteq \mathcal{H}_\ell \cap G = \{1\}$$

azpitalde $\mathbb{F}_p[[t]]$-analitikoz osaturiko G-ren serie subnormala da.

Deitu $H_i = \mathcal{H}_i \cap G$. Horrela, H_i bakoitza G-ren azpitalde $\mathbb{F}_p[[t]]$-analitikoa denez, H_{i-1}/H_i talde $\mathbb{F}_p[[t]]$-analitiko trinko abeldarra da $i \in \{2, \ldots, \ell\}$ guztietarako. Hortaz, Korolarioa 3.29ren ondorioz, $\mathrm{hspec}_{\mathrm{st}}(H_i/H_{i-1}) = [0,1]$ da. Lema 3.31 dela eta, $[\mathrm{hdim}_{\mathrm{st}}(H_i), \mathrm{hdim}_{\mathrm{st}}(H_{i-1})] \subseteq \mathrm{hspec}_{\mathrm{st}}(G)$ da $i \in \{2, \ldots, \ell\}$ guztietarako, eta, ondorioz, $\mathrm{hspec}_{\mathrm{st}}(G) = [0,1]$ da.

Jo dezagun azkenik kasu orokorrera. Korolarioa 3.33 dela eta, $Z(G)$ azpitalde $\mathbb{F}_p[[t]]$-analitikoa da eta, horrenbestez, Korolarioa 3.29 eta Lema 3.31ren arabera,

$$[0, \mathrm{hdim}_{\mathrm{st}}\, Z(G)] \subseteq \mathrm{hspec}_{\mathrm{st}}(G)$$

da. Bestetik, Proposizioa 1.59 eta Proposizioa 2.3ren ondorioz, $G/Z(G)$ talde $\mathbb{F}_p[[t]]$-analitiko trinkoa $\mathbb{F}_p[[t]]$-ren gainean lineala da. Hori dela eta, kasu lineala, Lema 3.21 eta Lema 3.22 uztartuz,

$$\mathrm{hspec}_{\{S_n Z(G)/Z(G)\}}(G/Z(G)) = \mathrm{hspec}_{\mathrm{st}}\,(G/Z(G)) = [0,1]$$

dugu, eta Korolarioa 3.12ren arabera,

$$[\mathrm{hdim}_{\mathrm{st}}\, Z(G), 1] \subseteq \mathrm{hspec}_{\mathrm{st}}(G),$$

da. Bereziki, espektroa $[0,1]$ tarte osoa da. $\square$

Horrenbestez, Lema 3.31 kontuan hartuta, talde $\mathbb{F}_p[[t]]$-analitiko baten espektroak tarte erreal bat barruan duela frogatzeko moduetako bat G-ren sekzio $\mathbb{F}_p[[t]]$-analitiko ebazgarri bat bilatzea da. Gure kasuan, bilaketa hori Titsen alternatibaren analogo topologikoan oinarrituko da. Baina lehenbizi ohartu honakoaz:

Lema 3.37. *Izan bedi G talde $\mathbb{F}_p[[t]]$-estandarra. Demagun hauetako bat betetzen dela:*

(i) *$Z(G)$ infinitua da edo*

(ii) *G-k ordena infinituko x elementu bat du.*

Orduan, $[0, {}^{1}/_{\dim G}] \subseteq \mathrm{hspec}(G)$ *da.*

Froga. Lehenbiziko baldintzapean, Korolarioa 3.33ren arabera, $Z(G)$ azpitalde $\mathbb{F}_p[[t]]$-analitiko abeldar infinitua da. Antzeko moduan, bigarren baldintzapean $Z(C_G(x))$ azpitalde $\mathbb{F}_p[[t]]$-analitiko abeldarra infinitua da, $\langle x \rangle \leq Z(C_G(x))$ baita.

Edozein kasutan G-k $H \leq G$ azpitalde $\mathbb{F}_p[[t]]$-analitiko abeldar infinitu bat du. Horrela, G trinkoa denez, H-k dimentsio analitiko hertsiki positiboa du, eta, Korolarioa 3.29ren arabera, H-ren espektro estandarra $[0,1]$ tartea da. Horrenbestez, Lema 3.31ren arabera,

$$[0, {}^{\dim H}/_{\dim G}] \subseteq \mathrm{hspec}_{\mathrm{st}}(G). \qquad \square$$

Teorema 3.38. *Izan bedi G talde $\mathbb{F}_p[[t]]$-analitiko trinkoa. Orduan,*

$$[0, {}^{1}/_{\dim G}] \subseteq \mathrm{hspec}_{\mathrm{st}}(G)$$

da.

Froga. Lehenik eta behin, Korolarioa 3.20gatik suposa dezagun orokortasunik galdu gabe G talde R-estandarra dela. Alde batetik, $Z(G)$ infinitua denean emaitza Teorema 3.37(i)en ondorioa da. Hortaz, suposa dezagun $Z(G)$ finitua dela. Orduan, $G/Z(G)$ talde $\mathbb{F}_p[[t]]$-analitikoak $\dim G$ dimentsioa du eta Korolarioa 3.11 dela eta,

$$\mathrm{hspec}_{\mathrm{st}}(G) = \mathrm{hspec}_{\mathrm{st}}(G/Z(G))$$

da. Bestalde, Proposizioa 2.3ren arabera, $G/Z(G)$ talde $\mathbb{F}_p[[t]]$-analitikoa $\mathbb{F}_p[[t]]$-ren gainean lineala da. Beraz, Titsen alternatiba topologikoa (ikusi [12, Teorema 1.3]) dela eta, $G/Z(G)$-k azpitalde ebazgarri ireki bat du, deitu H, edo azpitalde aske dentso bat. Lehenbiziko kasuan, H talde $\mathbb{F}_p[[t]]$-analitiko ebazgarria da eta $\dim G/Z(G) = \dim G$ dimentsioa du. Beraz, Teorema 3.36ren arabera,

$$\mathrm{hspec}_{\mathrm{st}}(G/Z(G)) = [0,1]$$

da. Bigarren kasuan, $G/Z(G)$-k ordena infinituko elementu bat du, eta emaitza Lema 3.37(ii)ren ondorioa da. $\square$

3.5 Chevalleyren talde klasikoak

Talde $\mathbb{F}_p[[t]]$-analitiko trinko baten Hausdorffen espektro estandarrak ez du zertan $[0,1]$ tarte osoa izan. Esate baterako, hartu $\mathrm{SL}_n(\mathbb{F}_p[[t]])$ talde lineal berezia. Jakina da $\mathrm{SL}_n(\mathbb{F}_p[[t]])$ taldea n^2-1 dimentsioko talde $\mathbb{F}_p[[t]]$-analitiko trinkoa dela, eta azpitalde $\mathbb{F}_p[[t]]$-estandar irekia hau duela

$$\mathrm{SL}_n^1(\mathbb{F}_p[[t]]) := \ker\{\mathrm{SL}_n(\mathbb{F}_p[[t]]) \to \mathrm{SL}_n(\mathbb{F}_p[[t]]/t\mathbb{F}_p[[t]])\}.$$

Horrela, [6, Korolarioa 1.5]en $\mathrm{SL}_2(\mathbb{F}_p[[t]])$-ren espektro estandarra guztiz zehazten da $p > 2$ denean, hau da,

$$\mathrm{hspec}_{\mathrm{st}}\left(\mathrm{SL}_2(\mathbb{F}_p[[t]])\right) = [0, 2/3] \cup \{1\}.$$

Gainera, [6, Teorema 1.4]n frogatzen denez, $p > 2$ denean

$$\mathrm{hspec}_{\mathrm{st}}\left(\mathrm{SL}_n(\mathbb{F}_p[[t]])\right) \cap \left(1 - \frac{1}{n+1}, 1\right) = \varnothing$$

da, alegia, 1 puntu isolatua da espektroan. Adibide horretan inspiratuz, espektro ez-osoa duten beste zenbait talde $\mathbb{F}_p[[t]]$-analitiko aurkeztuko ditugu. Zehazkiago, frogatuko dugu gainerako Chevalleyren talde klasikoetan ere 1 puntu isolatua dela espektroan. Horretarako,[6]ko ideiak erabiliko dira, eta Lieren aljebra graduatuan eginen da lan. Hasteko, talde horiek zertan datzaten azalduko dugu laburki. Dena den, funtsezko definizioetarako, erro sistemak kasu, edota gaia sakonago aztertu nahi izanez gero, irakurleak [15]era jo dezake.

Izan bedi R pro-p domeinua.

- R-ren gainean A_n $(n \geq 1)$ motako erro sistemari dagokion Chevalleyren talde klasikoa $\mathrm{SL}_{n+1}(R)$ da.
- R-ren gainean B_n $(n \geq 2)$ motako erro sistemak talde *talde ortogonal berezi* bakoitia definitzen du. Hots,

$$\mathrm{SO}_{2n+1}(R) := \left\{ A \in \mathrm{M}_{2n+1}(R) \;\middle|\; A^t K_{2n+1} A = K_{2n+1} \right\},$$

  non $K_n = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix} \in \mathrm{M}_n(R)$ den. Hori $n(2n+1)$ dimentsioko talde R-analitikoa da.

- R-ren gainean C_n $(n \geq 3)$ motako erro sistemak *talde sinplektikoa* definitzen du. Hots,

$$\mathrm{Sp}_{2n}(R) := \left\{ A \in \mathrm{M}_{2n}(R) \mid A^t J_{2n} A = J_{2n} \right\},$$

  non $J_{2n} = \begin{pmatrix} 0 & K_n \\ -K_n & 0 \end{pmatrix}$ den. Hori $n(2n+1)$ dimentsioko talde R-analitikoa da.

- R-ren gainean D_n ($n \geq 4$) motako erro sistemak *talde ortogonal berezi* bikoitia definitzen du. Hots,

  $$\mathrm{SO}_{2n}(R) := \left\{ A \in \mathrm{M}_{2n}(R) \mid A^t K_{2n} A = K_{2n} \right\}.$$

  Hori $n(2n-1)$ dimentsioko talde R-analitikoa da.

Talde horiek guztiak trinkoak dira, izan ere $\mathrm{M}_n(R) \cong R^{(n^2)}$ espazio topologiko trinkoaren azpimultzo itxiak dira. Alegia, Chevalleyren talde klasikoak R-ren gainean talde R-analitiko trinkoak dira. Halaber, beren elkarturiko aljebrak itxura hau du:

Teorema 3.39 (cf. [24, Ariketa 13.11(iii)])**.** *Izan bedi X_n erro sistema A_n ($n \geq 1$), B_n ($n \geq 2$), C_n ($n \geq 3$) edo D_n ($n \geq 4$) motakoa, eta izan bedi $G(R)$ X_n erro sistemari eta R pro-p domeinuari dagokion Chevalleyren talde klasikoa.*

(i) *$X_n = A_n$ bada, badago S talde R-estandar irekia non*

$$\mathcal{L}(S) \cong \mathfrak{sl}_{n+1}(R) = \{A \in \mathrm{M}_{n+1}(R) \mid \mathrm{tr}(A) = 0\}$$

den.

(ii) *$X_n = B_n$ bada, badago S talde R-estandar irekia non*

$$\mathcal{L}(S) \cong \mathfrak{so}_{2n+1}(R) = \left\{A \in \mathrm{M}_{2n+1}(R) \mid A^t = -A\right\}$$

den.

(iii) *$X_n = C_n$ bada, badago S talde R-estandar irekia non*

$$\mathcal{L}(S) \cong \mathfrak{sp}_{2n}(R) = \left\{A \in \mathrm{M}_{2n}(R) \mid J_{2n}A + A^t J_{2n} = 0\right\}$$

den (J_{2n} aurreko modu berean definituta dago).

(iv) *$X_n = D_n$ bada, badago S talde R-estandar irekia non*

$$\mathcal{L}(S) \cong \mathfrak{so}_{2n}(R) = \left\{A \in \mathrm{M}_{2n}(R) \mid A^t = -A\right\}$$

den. Hemen $\mathcal{L}(S)$-k S-ri elkarturiko Lieren aljebra adierazten du (konparatu Atala 1.4).

Lehenik eta behin, [54, Definizioa 2.9]ko objektua aurkeztuko dugu: (S, ϕ) talde R-estandarrari elkartutako $\mathrm{gr}\mathcal{L}(S) = \bigoplus_{n\geq 0} S_n/S_{n+1}$ *Lieren aljebra graduatua*. Aljebra horren Lieren kortxetea

$$[xS_{n+1}, yS_{m+1}]_{\mathrm{gr}\mathcal{L}(S)} := [x, y]S_{n+m+1}$$

araua bilinealki hedatuz lortzen da (eskuineko kortxeteek S taldeko kommutadorea adierazten dute).

Alde batetik, $[\cdot, \cdot]_{\mathrm{gr}\mathcal{L}(S)}$ Lieren kortxetea da. Izan ere, (1.5)en ondorioz,

$$\phi([x, y]) = \mathbf{B}(\phi(x), \phi(y)) - \mathbf{B}(\phi(y), \phi(x)) \mod \phi(S_{n+m+1})$$

da, eta, beraz, Lema 1.25 dela eta, $[\cdot, \cdot]_{\mathrm{gr}\mathcal{L}(S)}$-k Jacobiren identitatea betetzen du. Hots, R-Lie aljebra kortxetea da. Beste alde batetik, $x \in S_n$ eta $y \in S_m$ direnean, $[x, y] \in S_{n+m}$ da, eta, beraz, aurreko aljebra zenbaki arrunten gainean graduatua dago. Bestetik, izan bedi q $R/\mathfrak{m}$ hondar gorputzaren kardinalitatea. Horrela, S_n/S_{n+1} zatidura $R/\mathfrak{m}$-bektore espazioa da, eta, ondorioz, $\mathrm{gr}\mathcal{L}(S)$ $\mathbb{F}_q$-Lie aljebra graduatua da.

Edozein $H \leq_c S$ azpitalde itxik $\mathrm{gr}\mathcal{L}(S)$-ren azpialjebra graduatu bat definitzen du,

$$\mathrm{gr}\mathcal{L}(H) := \bigoplus_{n\geq 0} \frac{(H \cap S_n)S_{n+1}}{S_{n+1}}$$

(notazioa abusatuz $\mathrm{gr}\mathcal{L}(H)$ deituko dugu azpialjebra hori). Talde itxi guztiek azpialjebra bat definitzen duten arren, egon daitezke era honetan agertzen ez diren Lieren azpialjebrak ere.

Orokorrean, $L = \oplus_{n\geq 0} L_n$ graduatutako $\mathbb{F}_q$-Lie aljebran $K = \bigoplus_{n\geq 0} K_n$ graduatutako azpialjebraren *Hausdorffen dentsitatea* honela definitzen da:

$$\mathrm{hD}(K) := \liminf_{n\to\infty} \frac{\sum_{m\leq n} \dim_{\mathbb{F}_q} K_m}{\sum_{m\leq n} \dim_{\mathbb{F}_q} L_m}.$$

Halaber, aurreko definizioen arabera, $\mathrm{hD}(\mathrm{gr}\mathcal{L}(H)) = \mathrm{hdim}_{\{S_n\}}(H)$ da edozein

H azpitalde itxitarako. Izan ere,

$$\begin{aligned}
\mathrm{hD}(\mathrm{gr}\mathcal{L}(H)) &= \liminf_{n\to\infty} \frac{\sum_{m\le n} \dim_{\mathbb{F}_q} {}^{(H\cap S_m)S_{m+1}}/_{S_{m+1}}}{\sum_{m\le n} \dim_{\mathbb{F}_q} {}^{S_m}/_{S_{m+1}}} \\
&= \liminf_{n\to\infty} \frac{\sum_{m\le n} \log_q |H\cap S_m : H\cap S_{m+1}|}{\sum_{m\le n} \log_q |S_m : S_m + 1|} \\
&= \liminf_{n\to\infty} \frac{\log_q \prod_{m\le n} |H\cap S_m : H\cap S_{m+1}|}{\log_q \prod_{m\le n} |S_m : S_{m+1}|} \\
&= \liminf_{n\to\infty} \frac{\log_q |H : H\cap S_{n+1}|}{\log_q |S : S_{n+1}|} = \mathrm{hdim}_{\{S_n\}}(H). \qquad (3.9)
\end{aligned}$$

Hortaz, 1 puntua espektroan isolatua dela frogatzeko, $\mathrm{gr}\mathcal{L}(S)$-ren azpialjebra maximalei erreparatuko diegu. Konkretuago, azpialjebra baten Hausdorffen dentsitatea zehatz-mehatz 1 ez denean, dimentsio hori $1 - {}^1/_d$ balioarekin bornatuta dagoela ikusiko dugu. Horrenbestez, kodimentsio infinituko azpialjebrez arduratuko gara –kodimentsio finitukoek 1 Hausdorffen dentsitatea dute–. Horretarako $\mathrm{gr}\mathcal{L}(S)$-ren deskribapen hau dugu:

Lema 3.40 (cf. [24, Proposizioa 13.27] eta [54, Oharra 2.10 (5)])**.** *Izan bedi S talde R-estandarra. Orduan $R/\mathfrak{m}$-Lie aljebra gisa isomorfoak dira $\mathrm{gr}\mathcal{L}(S)$ eta $\mathcal{L}_0(S) \otimes_{R/\mathfrak{m}} \mathrm{gr}\,\mathfrak{m}$, non*

$$\mathcal{L}_0(S) = {}^{\mathcal{L}(S)}/_{\mathfrak{m}\mathcal{L}(S)} \qquad \text{eta} \qquad \mathrm{gr}\,\mathfrak{m} = \bigoplus_{n\ge 1} \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

diren.

Oharra. Ohartu $\mathrm{gr}\,\mathfrak{m}$ ideala $\mathrm{gr}(R)$ eraztun graduatuaren ideal maximala dela.

Froga. Lehenik, $\mathcal{L}_0(S)$ aljebra $R/\mathfrak{m}$-Lie aljebra gisa $(R/\mathfrak{m})^{(d)}$ da eta haren Lieren kortxetea $[\cdot,\cdot]_{\mathcal{L}}$ modulo $\mathfrak{m}$ murriztuta lortzen da. Beraz, $R/\mathfrak{m}$-bektore espazio gisa $\mathrm{gr}\mathcal{L}(S)$ eta $\mathcal{L}_0 \otimes_{R/\mathfrak{m}} \mathrm{gr}\,\mathfrak{m}$ elkarri isomorfoak dira. Gainera, $[\cdot,\cdot]_{\mathcal{L}}$ bilineala denez eta tentsore biderketak bilialtasuna gordetzen duenez, aurreko isomorfismoa $R/\mathfrak{m}$-Lie aljebra isomorfismoa da. □

Bereziki, $R = \mathbb{F}_p[[t]]$ denean, Lieren aljebra graduatua $\mathcal{L}_0(S) \otimes_{\mathbb{F}_p} t\mathbb{F}_p[t]$ da. Izan bitez orain F gorputza eta $\mathcal{G}$ dimentsio finituko F-aljebra perfektua, hau da, $[\mathcal{G},\mathcal{G}] = \mathcal{G}$ da. Orduan, $\mathcal{G} \otimes_F tF[t]$ Lieren F-aljebra graduatua defini dezakegu, $[A\otimes t^n, B\otimes t^m] := [A,B]_{\mathcal{G}} \otimes t^{n+m}$ izanik Lieren kortxetea tentsore elementalen gainean. Ohartu honakoaz:

Lema 3.41. *Izan bedi aurreko $\mathcal{L} = \mathcal{G} \otimes_F tF[t]$ F-Lie aljebra. Orduan, kodimentsio infinituko F-azpialjebra graduatu oro kodimentsio infinitua edukitzearekiko maximala den F-azpialjebra graduatu baten parte da.*

Froga. Lehenik eta behin, $\mathcal{L}$ finituki sortutako F-azpialjebra bat da. Izan ere, $\{x_1, \ldots, x_m\}$ multzoa $\mathcal{G}$ aljebraren sistema sortzailea bada, $S = \{x_1, \ldots, x_m, x_1 \otimes t, \ldots, x_m \otimes t\}$ multzoak $\mathcal{L}$ sortzen du. Azken batean, $\langle S \rangle_F$, S-k sortutako Lie aljebrak, barruan ditu $\mathcal{G}$ eta $\mathcal{G} \otimes t$. Demagun indukzioz $\mathcal{G} \otimes t^{n-1} \subseteq \langle S \rangle_F$ dela. Orduan, $\mathcal{G}$ perfektua denez,

$$\mathcal{G} \otimes t^n = [\mathcal{G}, \mathcal{G}] \otimes t^n = \left[\mathcal{G} \otimes t^{n-1}, \mathcal{G} \otimes t\right] \subseteq \langle S \rangle_F$$

da.

Azkenik emaitza Zorneren Lemaren ondorioa da. Ikusi behar dugu kodimentsio infinituko F-azpialjebra graduatuen kate guztiek partekotasunarekiko elementu maximal bat dutela. Kodimentsio infinituko F-azpialjebra graduatuak partekotasunarekiko partzialki ordenatuta daude, eta hartu totalki ordenatutako $\{H_i\}_{i \in I}$ azpimultzoa. Definitu $H = \cup_{i \in I} H_i$ F-azpialjebra graduatua. Demagun absurdura eramanez, H-k kodimentsio finitua duela $\mathcal{L}$-n, eta, beraz, finituki sortutako F-Lie azpialjebra dela, hau da, $H = \langle h_1, \ldots, h_r \rangle_F$. Orduan, existitzen da $i_0 \in I$ non $h_k \in H_{i_0}$ den $k \in \{1, \ldots, r\}$ guztietarako, beraz, $H = H_{i_0}$ F-aljebra graduatuak kodimentsio infinitua du $\mathcal{L}$-n, hau da, kontraesana dugu. Hortaz, $\{H_i\}_{i \in I}$ kateak partekotasunarekiko elementu maximal bat du kodimentsio infinituko F-azpialjebra graduatuen familian. □

Hurrengo emaitzari esker, kodimentsio maximala izatearekiko maximalak diren azpialjebra graduatuen dentsitatea borna dezakegu.

Teorema 3.42 (cf. [6, Korolarioa 5.3])**.** *Izan bitez F gorputza, $\mathcal{G}$ F-aljebra bakun zentrala eta $\mathcal{L} = \mathcal{G} \otimes_F tF[t]$. Izan bedi $\mathcal{K}$ kodimentsio infinitua izatearekiko maximala den $\mathcal{L}$-ren F-azpialjebra graduatua. Orduan, $\mathcal{K}$-ren Hausdorffen dentsitatea $1/q$ non q zenbaki lehen bat den, edo $\dim_F \mathcal{H} / \dim_F \mathcal{G}$ da, non $\mathcal{H}$ $\mathcal{G}$-ren azpialjebra graduatu maximal bat den.*

Oharra 3.43. Gogoratu dimentsio finituko $\mathcal{G}$ F-aljebra *bakun zentrala* dela bakuna bada eta haren *zentroidea*, i.e.

$$\mathrm{Cent}(\mathcal{G}) = \{f \in \mathrm{End}_F(\mathcal{G}) \mid f([x, y]) = [f(x), y] \ \forall x, y \in \mathcal{G}\},$$

F-ri isomorfoa bada. Haatik, F finitua bada, aurreko teoremaren baldintzak erlaxatu daitezke, eta nahikoa da $\mathcal{G}$ F-aljebra bakuna izatea. Izan ere, aurreko teorema [7, Teorema 4.1]eko modu berean frogatzen da, eta autoreek [7, Teorema 1.1]en ondoko oharrean dioten moduan, F finitua denean, nahikoa da $\mathcal{G}$ bakuna izatea.

Elkar ditzagun emaitza horiek guztiak azpiatal honetan bila genbiltzan emaitza frogatzeko.

Korolarioa 3.44. *Izan bedi X_n erro sistema A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$ edo D_n $(n \geq 4)$ motakoa eta izan bedi $G = G(X_n)$ X_n-ri $\mathbb{F}_p[[t]]$-ren gainean elkarturiko Chevalleyren talde klasikoa. Halaber, izan bedi $L(Q)$ erro sistema horri elkarturiko Lieren aljebra Q eraztunaren gainean. Demagun $L(\mathbb{F}_p)$ bakuna dela, orduan*

$$\mathrm{hspec}_{\mathrm{st}}(G) \cap \left(1 - \frac{1}{\dim G}, 1\right) = \varnothing$$

da.

Froga. Alde batetik, Teorema 3.39ren arabera, G-k S azpitalde $\mathbb{F}_p[[t]]$-estandar ireki bat du zeinari elkarturiko Lieren aljebra $L(\mathbb{F}_p[[t]])$-ri isomorfoa den. Bestalde, Lema 3.40 dela eta, $\mathrm{gr}\mathcal{L}(S) \cong \mathcal{L}_0 \otimes_{\mathbb{F}_p} \mathrm{gr}\,\mathfrak{m}$ da $\mathbb{F}_p$-bektore espazio gisa, non

$$\mathcal{L}_0 = L(\mathbb{F}_p[[t]])/tL(\mathbb{F}_p[[t]]) \cong L(\mathbb{F}_p) \quad \text{eta} \quad \mathrm{gr}\,\mathfrak{m} = \bigoplus_{n \geq 1} (t^n) \,/\, \left(t^{n+1}\right)$$

diren. Hots, $\mathrm{gr}\mathcal{L}(S) \cong L(\mathbb{F}_p) \otimes_{\mathbb{F}_p} t\mathbb{F}_p[t]$ da.

Beste alde batetik, demagun $H \leq_c S$ azpitalde itxiak $\mathrm{hdim}_{\mathrm{st}}(H) < 1$ betetzen duela. Orduan, $|S : H|$ infinitua da eta $\mathrm{gr}\mathcal{L}(H)$-k kodimentsio infinitua du $\mathrm{gr}\mathcal{L}(S)$-n. Horrela, $L(\mathbb{F}_p)$ bakuna denez, Lema 3.41en arabera, $\mathrm{gr}\mathcal{L}(H)$ $\mathbb{F}_p$-azpialjebra graduatua $\mathrm{gr}\mathcal{L}(S)$-ren $\mathcal{M}$ $\mathbb{F}_p$-azpialjebra graduatu baten barruan dago, $\mathcal{M}$ kodimentsio infinitua izatearekiko maximala izanik. Hortaz, Teorema 3.42, Oharra 3.43 eta identitatea (3.9) direla eta:

$$\begin{aligned}\mathrm{hdim}_{\mathrm{st}}(H) &= \mathrm{hD}(\mathrm{gr}\mathcal{L}(H)) \leq \mathrm{hD}(\mathcal{M}) \\ &\leq \max\left\{\frac{1}{2}, \frac{\dim_{\mathbb{F}_p} \mathcal{H}}{\dim_{\mathbb{F}_p} L(\mathbb{F}_p)} \;\middle|\; \mathcal{H} \text{ azpialjebra maximala da } L(\mathbb{F}_p)\text{-n}\right\} \\ &\leq 1 - \frac{1}{\dim_{\mathbb{F}_p} L(\mathbb{F}_p)} = 1 - \frac{1}{\dim S},\end{aligned}$$

$\dim S = \dim G(X_n) = \dim_{\mathbb{F}_p} L(\mathbb{F}_p)$ baita (konparatu Teorema 3.39). Azkenik, Korolarioa 3.20ren arabera, $\mathrm{hspec}_{\mathrm{st}}(G) = \mathrm{hspec}_{\mathrm{st}}(S)$ da eta badugu emaitza. □

Azkenik, karakteristika positiboko gorputz finituen gainean, $\mathfrak{so}_n(F)$ eta $\mathfrak{sp}_{2n}(F)$ motako aljebrak sakonki aztertu izan dira. Hala, $p = 2$ denean, horietako bakarra ere ez da bakuna (ikusi [70, Atala 4.4]), baina $p \geq 3$ denean, jakina da $\mathfrak{so}_n(F)$ $(n \geq 5)$ eta $\mathfrak{sp}_{2n}(F)$ $(n \geq 2)$ aljebra bakunak direla (ikusi [70, 181. orria eta Atala 4.4]). Hori dela eta, honakoa ondoriozta dezakegu:

Korolarioa 3.45. *Izan bedi $p \geq 3$ eta demagun G taldea $\mathrm{SO}_n(\mathbb{F}_p[[t]])$ $(n \geq 5)$ edo $\mathrm{Sp}_{2n}(\mathbb{F}_p[[t]])$ $(n \geq 2)$ dela. Orduan,*

$$\mathrm{hspec}_{\mathrm{st}}(G) \cap \left(1 - \frac{1}{\dim G},\ 1\right) = \varnothing$$

da.

3.6 Oharrak

Espazio euklidearretan geometria fraktalaren inguruko erreferentzia behinena Falconerren [26] liburua da. Azpimarratu behar da bertan $\mathbb{R}^{(n)}$-n egiten dela lan, baina emaitzak orokortasun guztiarekin zein-nahi espazio metrikotan betetzen dira; eta kapituluan zehar eman diren erreferentziak onargarriak dira hori kontuan hartuta.

Atala 3.2k [27] darraio, ideia gehienak bertatik hartuta daude, gure kasuan azpibarietateekin, eta ez azpitaldeekin, jarduteak sortutako zailtasun tekniko batzuk salbu.

Azkenik, Atala 3.3tik 3.5era bitarteko emaitzak eskuarki originalak dira, eta [30]en argitaratu dira, González-Sánchezekin elkarlanean. Halere, Atala 3.5eko teknikak Barnea eta Shalevenak [6] dira. Orobat, aipatu behar da Teorema 3.3 talde R-analitiko *guztietara* orokortzeko oztopo nagusia, alta ez bakarra, hau dela: talde R-analitiko trinkoetan oraindik ez dago azpitalde R-analitikoak topatzeko irizpide sinplerik, hots, ez dago Proposizioa 1.53ren pareko emaitzarik.

Azkenik, doktoretzak iraun bitartean, de las Herasekin batera [36] euskarazko dibulgazio artikulua argitaratu da dimentsio fraktalei buruz.

Hasieran bazen Hitza

Joan 1:1

4
Hitzak talde R-analitiko trinkoetan

HITZ bat k aldagaitan k sortzaileko $F(x_1, \ldots, x_k)$ talde askeko $w(x_1, \ldots, x_k)$ elementua da, hau da,

$$x_{i_1}^{\varepsilon_{i_1}} \ldots x_{i_m}^{\varepsilon_{i_m}},$$

non $m \in \mathbb{N}_0$, $i_j \in \{1, , \ldots, k\}$ eta $\varepsilon_{i_j} = \pm 1$ diren; $m = 0$ deneko kasu bereziari *hitz hutsa* deritzo.

Behin G talde bat finkatuta, $w(x_1, \ldots, x_k)$ hitza $w\colon G^{(k)} \to G$ funtzioa balitz bezala ikus daiteke, $w(g_1, \ldots, g_k)$ irudia i guztietarako x_i-ren order g_i jarrita lortzen den G-ko elementua delarik. Funtzio horri w-ren *hitz funtzioa* deitzen zaio. Hitzak bigarren modu horretan ulertzea komeni da, hau da, bi hitz *baliokideak* dira G talde guztietan hitz funtzio bera definitzen badute. Horrela, agerikoa da bi hitz baliokideak direla baldin eta soilik baldin talde askean forma murriztu bera hartzen badute.

Definizioa 4.1. Izan bitez G taldea eta $w(x_1, \ldots, x_k)$ hitza. *Hitz balioen* edo *w-balioen* multzoa G-n

$$w\{G\} := \{w(g_1, \ldots, g_k) \mid g_i \in G\}$$

da, edo, baliokideki, $\operatorname{im} w$ irudia w funtzio gisa badakusagu.

Lema 4.2. *Izan bitez G talde abeldarra eta w hitza. Orduan, $w\{G\}$ azpitaldea da.*

Froga. Taldea abeldarra denez,

$$w(g_1, \ldots, g_k) \cdot w(h_1, \ldots, h_k) = w(g_1h_1, \ldots, g_kh_k) \in w\{G\},$$

eta $w(g_1, \ldots, g_k)^{-1} = w\left(g_1^{-1}, \ldots, g_k^{-1}\right) \in w\{G\}$ dira $g_i, h_i \in G$ guztietarako. $\square$

Dena den, $w\{G\}$ ez da orohar G-ren azpitaldea, eta w-ren *hitzezko azpitaldea* G-n

$$w(G) := \langle w\{G\} \rangle \tag{4.1}$$

moduan definitzen da –nahiz eta eskuizkribu honetan zehar (4.1) definizioa mantenduko dugun, talde topologikoetan $w(G)$ hitz balioek sortutako talde abstraktuaren itxitura topologikoa gisa definitzen da askotan–. Hurrengo emaitza sarritan erabiliko dugu erreferentziarik eman gabe:

Lema 4.3. *Izan bitez w hitza eta $\varphi\colon G \to H$ talde homomorfismoa. Orduan, $\varphi\left(w(G)\right) = w\left(\varphi\left(G\right)\right)$ da. Bereziki, $w(G) \operatorname{char} G$ da eta $N \trianglelefteq G$ denean,*

$$w\left(G/N\right) = w(G)N/N$$

da.

Froga. Ohartu $\varphi(w(g_1, \ldots, g_k)) = w\left(\varphi(g_1), \ldots, \varphi(g_k)\right)$ dela $g_i \in G$ guztietarako. $\square$

Definizioa 4.4. Izan bitez G taldea eta w hitza. Orduan, w-ren *azpitalde marjinala* G-n

$$w^*(G) := \left\{ g \in G \;\middle|\; \begin{array}{c} w(x_1, \ldots, x_k) = w\left(x_1, \ldots, x_{i-1}, x_ig, x_{i+1}, \ldots, x_k\right) \\ \forall i \in \{1, \ldots, k\},\ \forall x_j \in G \end{array} \right\}$$

da. Bestalde, $H \leq G$ *marjinala* da w-rentzat $H \leq w^*(G)$ bada.

Bereziki, $H \leq G$ marjinala denean w-rentzat, $w(H) = \{1\}$ betetzen da. Gainera, $g \in w^*(G)$ bada, $w(x_1, \ldots, x_k) = w(x_1, \ldots, gx_i, \ldots, x_k)$ da $i \in \{1, \ldots, k\}$ eta $x_j \in G$ guztietarako, hau da, definizioan g ezkerrean edo eskuinean jartzearen antisimetria itxurazkoa baino ez da. Izan ere, hartu $g \in G$ elementua eta demagun $w(x_1, \ldots, x_i g, \ldots, x_k) = w(x_1, \ldots, x_k)$ dela $x_j \in G$ guztietarako, orduan

$$\begin{aligned} w(x_1, \ldots, gx_i, \ldots, x_k) &= w(x_1, \ldots, x_i g^{x_i}, \ldots, x_k) \\ &= w\left(x_1, \ldots, (x_i g)^{x_i}, \ldots, x_k\right) \\ &= w\left(x_1^{x_i^{-1}}, \ldots, x_i g, \ldots, x_k^{x_i^{-1}}\right)^{x_i} \\ &= w\left(x_1^{x_i^{-1}}, \ldots, x_i, \ldots, x_k^{x_i^{-1}}\right)^{x_i} = w(x_1, \ldots, x_i, \ldots, x_k) \end{aligned}$$

da.

Adibideak 4.5. Izan bedi G taldea.

(i) G talde totala eta $\{1\}$ azpitalde tribiala hitzezko azpitaldeak dira, hurrenez huren, $w(x) = x$ eta hitz hutsari dagozkionak.

(ii) Hitz arrunt eta ezagunena *kommutadore hitza* da, hau da, $\gamma_2(x, y) = [x, y] = x^{-1}y^{-1}xy$. Haren hitzezko azpitaldea $\gamma_2(G) = G'$ *azpitalde deribatua* da eta dagokion azpitalde marjinala $\gamma_2^*(G) = Z(G)$ zentroa.

(iii) *Hitz behe zentralak* errekurtsiboki

$$\gamma_n(x_1, \ldots, x_n) := [\gamma_{n-1}(x_1, \ldots, x_{n-1}), x_n] \ \forall n \geq 3$$

gisa definitzen dira, eta *hitz deribatuak* errekutsiboki $\delta_1(x_1, x_2) := \gamma_2(x_1, x_2)$ eta

$$\delta_n(x_1, \ldots, x_{2^n}) := [\delta_{n-1}(x_1, \ldots, x_{2^{n-1}}), \delta_{n-1}(x_{2^{n-1}+1}, \ldots, x_{2^n})] \ \forall n \geq 2$$

moduan definitzen dira.

(iv) *Burnsideren hitzak* $w_m(x) = x^m$ dira. Horiek G^m hitzezko azpitaldeak definitzen dituzte, hau da, G-ko elementuen $m$garren berreturek sorturiko taldeak. Eta, adibidez, $w_2^*(G)$ azpitalde marjinala gehienez 2 ordenako elementu zentralek osatzen dute, hau da, $w_2^*(G) = \{g \in Z(G) \mid g^2 = 1\}$ da.

P. Hallek [33] hainbat galdera egin zituen w-balioen multzoaren eta haren hitzezko azpitaldearen eta azpitalde marjinalaren arteko erlazioaren inguruan. Hurrengo definizioak itaun horiek laburtzeko balioko du:

Definizioa 4.6. Izan bitez w hitza eta $\mathcal{C}$ talde klasea.

(i) w hitza *laburra* da $\mathcal{C}$-n, $G \in \mathcal{C}$ guztietarako $w\{G\}$ finitua izateak $w(G)$ ere finitua dela inplikatzen badu.

(ii) w hitza *sendoa* da $\mathcal{C}$-n, $G \in \mathcal{C}$ guztietarako $|G : w^*(G)|$ finitua izateak $w(G)$ finitua dela inplikatzen badu.

Horrela, w *laburra* (*sendoa*) da talde guztien klasean laburra (sendoa) denean. Antzeko moduan, G taldean hitz guztiak laburrak badira, G *hitzez laburra* dela diogu.

Orokorrean, $|w\{G\}| \leq |G : w^*(G)|^k$ denez (k zenbaki osoa w hitzaren aldagai kopurua da), sendotasuna laburtasuna baino gogorragoa da: w laburra bada $\mathcal{C}$-n, orduan w sendoa da $\mathcal{C}$-n. Haatik, talde erresidualki finituetan, eta guri ardura zaizkigun taldeak horrelakoak dira, bi kontzeptuak baliokideak dira:

Lema 4.7 (cf. [67, Lema 1.4.1])**.** *Izan bitez G taldea eta w hitza.*

(i) *$|G : w^*(G)|$ finitua bada, orduan $w\{G\}$ finitua da.*

(ii) *G erresidualki finitua eta $w\{G\}$ finitua badira, orduan $|G : w^*(G)|$ finitua da.*

P. Hallek bat hitz guztiak laburrak zirela iragarri zuen. Aitzitik, ia hiru hamarkadaren ostean, Ivanovek [39] aieru hori errefuxatu zuen, G talde bat eta w hitz bat topatu baitzituen non $w\{G\}$ multzoak bi elementu dituen, baina $w(G)$ talde zikliko infinitua den. Hala ere, kontradibide hori ez da erresidualki finitua, ezta Ol'shanskiĭk (ikusi [62, Teorema 39.7]) eraiki zuen antzeko kontradibidea ere. Horrek Jaikin-Zapirainek [44] eta Segalek [67] proposaturiko aieru honetara garamatza:

Aierua 4.8 (Laburtasunaren aierua talde erresidualki finituetan)**.** Hitz guztiak laburrak dira talde erresidualki finituen klasean.

Talde hitzez laburren klase gutxi batzuk baino ez dira ezagutzen. Talde abeldarren (ikusi Lema 4.2) eta periodikoen (ikusi datorren Lema 4.15) ageriko adibideez gain; 1960ko hamarkadan, Merzjalkovek [57] eta Turner-Smithek [72] hurrenez hurren frogatu zuten talde linealak eta zatidura guztiak erresidualki finituak dituzten taldeak (e.g. talde birtualki nilpotenteak) hitzez laburrak direla.

Hitz balioen multzoa infinitua denean, laburtasunaren pareko kontzeptua *hitz eliptikotasuna* da. Hori definitzeko notazio hau erabiliko da: $X \subseteq G$ azpimultzo baterako, izendatu $X^{*\ell}$ moduan $X \cup X^{-1} \cup \{1\}$ multzoko ℓ elementuren biderketek osatzen duten multzoa.

Definizioa 4.9. Izan bitez G taldea eta w hitza. Orduan, w *eliptikoa* da G-n existitzen bada $\ell \in \mathbb{N}$ non $w(G) = w\{G\}^{*\ell}$ den.

Aurreko baldintza betetzen duten ℓ zenbaki osoetan txikienari w-ren *hitz zabalera* deritzo. Horrela, G taldea *hitzez eliptikoa* da hitz guztiak G-n eliptikoak direnean. Eliptikotasuna laburtasuna baino gogorragoa da: w eliptikoa bada $\mathcal{C}$ talde klaseko talde guztietan, orduan w laburra da $\mathcal{C}$-n.

Lema 4.2ren arabera, talde abeldarrak hitzez eliptikoak dira eta 1 hitz zabalera dute. Horiez gain, talde aljebraiko linealak* (ikusi [56]), finituki sortutako talde abeldar-bider-nilpotenteak (ikusi [29] eta [71]) edo, tesi honen gaiarekin zerikusi zuzena duena, talde p-adiko analitiko trinkoak (ikusi [44]) hitzez eliptikoak dira.

Haatik, hitzez eliptikoak ez diren taldeen adibide naturalak daude (kontradibideak ez dira hitzezko laburtasunarenak bezain konplexuak bederen). Esate baterako, Roman'kovek [66] finituki sortutako pro-p talde ebazgarri bat aurkeztu zuen non $\delta_2(x_1, \ldots, x_4) = [[x_1, x_2], [x_3, x_4]]$ hitz deribatuak zabalera infinitua duen.

Talde profinituei dagokionez, hitz zabalera eta hitzezko azpitaldea itxia izatea lotuta daude.

Proposizioa 4.10. *Izan bitez G Hausdorff talde topologiko trinkoa eta w hitza. Orduan, w eliptikoa da G-n baldin eta soilik baldin $w(G)$ itxia bada.*

Froga. Soilik baldin norantzan, ohartu edozein n-tarako $w\{G\}^{*n}$ itxia dela, multzo trinko baten irudi jarraitua baita. Hortaz, w-ren hitz zabalera ℓ bada, $w(G) = w\{G\}^{*\ell}$ itxia da.

Bestetik *baldina* frogatzeko, ohartu

$$w(G) = \bigcup_{n \in \mathbb{N}} w\{G\}^{*n}$$

dela eta $w\{G\}^{*n}$ guztiak itxiak direla. Hori dela eta, $w(G)$ Hausdorff eta trinkoa denez, Baireren Kategoria Teoremaren (ikusi [59, Teorema 48.2]) arabera, existitzen da m zenbaki osoa non $w\{G\}^{*m}$-k barnealde ez-hutsa duen, hau da, $U \subseteq_o w(G)$ azpimultzo ireki ez-huts bat du barruan. Hortaz,

$$w(G) = \bigcup_{g \in w(G)} gU$$

***talde aljebraiko lineal* diogunean, $\mathrm{GL}_n(K)$-ren azpitalde Zariski itxi bat esan nahi dugu, K gorputz aljebraikoki itxia delarik.

da, eta $w(G)$-ren trinkotasunagatik

$$w(G) = \bigcup_{i=1}^{r} g_i U$$

da $g_1, \dots, g_r \in w(G)$ elementu batzuetarako. Hartu $k \in \mathbb{N}$ non $g_i \in w\{G\}^{*k}$ den $i \in \{1, \dots, r\}$ guztietarako, orduan

$$w(G) = \bigcup_{i=1}^{r} g_i U \subseteq w\{G\}^{*(k+m)},$$

da, nahi genuen moduan. □

Orokorrean, (hitzezko) azpitalde bat itxia den ala ez jakitea erabilgarria da talde profinituekin jarduterakoan. Horregatik, Jaikin-Zapirainen emaitza nabarmen hau enuntziatu behar dugu:

Teorema 4.11 (cf. [44, Teorema 1.1])**.** *Izan bedi $w \in F_k$ hitza k aldagaitan. Orduan, w hitzak zabalera finitua finituki sortutako pro-p talde guztietan baldin eta soilik baldin $w \notin \delta_2(F_k)\,(F_k')^p$ bada.*

Kapitulu honen xedea talde R-analitiko trinkoak hitzez laburrak direla frogatzea da. Aipatu beharrekoa da char $R = 0$ denean, emaitza hori Teorema 2.27ren –talde R-analitiko trinkoak linealak dira– eta Merzjalkoven Teoremaren –talde linealak hitzez laburrak dira– ondorio zuzena dela. Aitzitik, interesgarria da horren froga independentea ematea, zeina pro-p domeinu guztietarako, karakteristika edozein delarik ere, betetzen den.

Are gehiago, emaitza orokor hau talde R-analitiko trinko *guztiak* linealak izatearen aldeko beste ebidentzia bat da.

4.1 Laburtasuna talde R-estandarretan

Talde R-estandarren klasean laburtasuna berehalakoa da:

Proposizioa 4.12. *Izan bitez S talde R-estandarra eta w hitza. Demagun $w\{S\}$ finitua dela, orduan $w(S) = \{1\}$ da.*

Froga. Lehenik eta behin, S taldea $\left(\mathfrak{m}^N\right)^{(d)}$-rekin identifikatu daiteke, non N taldearen maila eta d dimentsioa diren. Horrela, biderketa eta alderantzizkoa bi berretura serie formalen tuplak definitzen dituzte eta eta identitatea $\mathbf{0}$ da. Horrenbestez, w hitz funtzioa $\mathbf{W} \in R[[X_1, \dots, X_{dk}]]^{(d)}$ berretura serie tupla bakarra da (k hitzeko indeterminatu kopurua da). Horrela, $w\{S\}$ finitua eta hitz funtzioa jarraitua direnez, $\mathbf{W}$ lokalki konstantea da; eta, beraz, Lema 1.8ren arabera, $\mathbf{W}$ konstantea da. Hots, $\mathbf{W}(X_1, \dots, X_{dk}) = \mathbf{W}(\mathbf{0}, \dots, \mathbf{0}) = \mathbf{0}$ da, eta, beraz, $w\{S\} = \{\mathbf{0}\}$. □

Emaitza horren pare bat ondorio aipatu behar ditugu. Alde batetik, talde R-analitiko guztiek laburtasunaren aieruaren bertsio ahulago hau betetzen dute:

Korolarioa 4.13. *Izan bitez G talde R-analitikoa eta w hitza. Demagun $w\{G\}$ finitua dela. Orduan, existitzen da S azpitalde R-estandar irekia non w legea den, hau da, $w(S) = \{1\}$ da.*

Froga. Lema 1.21en arabera, badago S azpitalde R-estandar ireki bat G-n. Horrela, $|w\{S\}| \leq |w\{G\}|$ denez, Proposizioa 4.12 dela eta, $w(S) = \{1\}$ da. □

Beste alde batetik, G talde R-analitiko trinkoa bada eta $w\{G\}$ finitua, w-balioen multzoa soilik azpitalde R-estandar jakin baten ezker koklaseei begira kalkula daiteke. Hots, izan bitez G talde R-analitiko trinkoa eta $S \trianglelefteq_o G$ azpitalde R-estandar irekia zeinaren konjokazio funtzioak hertsiki analitikoak diren (bigarrena Lema 1.23 dela eta existitzen da R ez denean IND bat), eta izan bedi T ezker transbertsal bat S-rentzat G-n. Oroitu S-tik eratorritako atlasa, hau da, $\{(tS, \phi_t)\}_{t \in T}$ non $\phi_t(x) = \phi(t^{-1}x)$ den. Lema 1.24ren ondorioz, $w\colon G^{(k)} \to G$ funtzio R-analitikoa $t_1S \times \cdots \times t_kS$ ($t_i \in T$) multzo irekian berretura serie tupla bakarrak emanda dago –w ez da biderketa eta alderantzizko funtzioen konposaketa egokia besterik–. Alegia, existitzen da $\mathbf{W}_{t_1,\dots,t_k} \in R[[X_1, \dots, X_{dk}]]^{(d)}$ berretura serie formalen tupla non

$$\phi_p(w(x_1, \dots, x_k)) = \mathbf{W}_{t_1,\dots,t_k}\left(\phi_{t_1}(x_1), \dots, \phi_{t_k}(x_k)\right) \ \forall x_j \in t_jS \tag{4.2}$$

den, hemen p elementua $w(t_1, \dots, t_k)p^{-1} \in S$ betetzen duen T-ko elementu bakarra da.

Gainera, $w\{G\}$ finitua bada, w funtzio jarraitua lokalki konstantea da, eta Lema 1.8ren eraginez, $\mathbf{W}_{t_1,\dots,t_k}$ konstantea da, hau da,

$$\mathbf{W}_{t_1,\dots,t_k}(X_1, \dots, X_{dk}) = \mathbf{c} \in R^{(d)}.$$

Hots, $\phi_p(w(x_1, \dots, x_k)) = \mathbf{c}$ da $x_j \in t_jS$ guztietarako. Beste era batera esanda:

Proposizioa 4.14. *Izan bitez w hitza, G talde R-analitiko trinkoa eta S azpitalde normal R-estandarra zeinaren konjokazio funtzioak hertsiki analitikoak diren. Orduan, $w\{G\}$ finitua bada, S marjinala da w-rentzat.*

Frogapen teknika hasierako problema bat Krull dimentsioko pro-p domeinu baten gainean analitikoa den talde batera murriztean datza, eta horretarako Atala 2.2n deskribaturiko eraztun aldaketa erabiliko da. Hori dela eta, lehenbiziko bat dimentsioko kasua aztertu behar da. Orobat, talde p-adiko analitiko trinkoak hitzez laburrak dira, linealak baitira Korolarioa 2.6ren arabera. Hortaz, Korolarioa 1.44 kontuan hartuta, $R = \mathbb{F}_p[[t]]$ kasura murriz gaitezke.

Hainbat emaitza tekniko erabiliko ditugu. Horietako batzuk frogatuko dira, baina beste batzuk enuntziatu baino ez ditugu eginen. Lehenik, gogora dezagun emaitza ezagun hau:

Lema 4.15. *Izan bitez G taldea eta w hitza. Demagun $w\{G\}$ finitua dela. Orduan, $w(G)'$ finitua da, eta $w(G)$ finitua da baldin eta soilik baldin w-balio guztiek ordena finitua badute G-n.*

Froga. Izan bedi $g \in G$. Lema 4.3ren arabera, $w\{G\}^g \subseteq w\{G\}$ da, hau da, $x \in w\{G\}$ guztietarako x^G konjokazio klasea $w\{G\}$-n dago. Hortaz,

$$|G : C_G(x)| = \left|x^G\right| \leq |w\{G\}|$$

da, eta $C_G(x)$-k indize finitua du G-n. Horrenbestez, $C_G(w(G)) = \cap_{x\in w\{G\}} C_G(x)$ azpitaldeak indize finitua du G-n, eta, ondorioz, $|w(G) : Z(w(G))|$ ere finitua da. Beraz, Schurren Teoremaren arabera (ikusi [65, Teorema 10.1.4]), $w(G)'$ finitua da.

Azkenik, $w(G)$ finitua bada, exponente finitua eduki behar du. Alderantziz, demagun $w\{G\}$-ko elementuek ordena finitua dutela, orduan $w(G)/w(G)'$ talde abeldarra ordena finituko elementu kopuru finitu batek sortzen du, bereziki, finitua da; eta $w(G)'$ finitua denez emaitza erdiesten dugu. □

Schur motako emaitza hau ere beharko dugu:

Lema 4.16 (cf. [45, Proposizioa 5.1])**.** *Izan bitez G taldea eta N azpitalde normal nilpotentea. Demagun $\frac{NZ(G)}{Z(G)}$ zatidurak exponente finitua duela. Orduan, $[N, G]$ azpitaldeak exponente finitua du.*

Froga. Hall-Petrescuren formularen arabera (ikusi [38, III.9.4]), $m \in \mathbb{N}$ guztietarako

$$x^m y^m = (xy)^m c_2(x,y)^{\binom{m}{2}} \dots c_m(x,y)^{\binom{m}{m}} \tag{4.3}$$

da, non $c_r(x,y) \in \gamma_r(\langle x,y\rangle)$ den.

Izan bedi m zenbakia $\frac{NZ(G)}{Z(G)}$ zatidura taldearen exponentea, orduan (4.3) dela eta, $n \in N$ eta $g \in G$ guztietarako:

$$[n,g]^m \equiv n^{-m}(n[n,g])^m = n^{-m}(n^g)^m = [n^m,g] = 1 \mod \gamma_2(K) \tag{4.4}$$

da, non $K := \langle n,[n,g]\rangle \leq N$ den. Are gehiago, (4.3) eta (4.4) direla eta, $l \in \mathbb{N}$ guztietarako

$$([n_1,g_1]\dots[n_l,g_l])^m \equiv [n_1,g_1]^m \dots [n_l,g_l]^m \equiv 1 \mod \gamma_2(N)$$

da.

Izan bedi $\eta(m)$ zenbakia 2 sortzaileko eta m exponenteko talde nilpotente handienaren ordena†. Nahikoa da $\gamma_2(N)^{\eta(m)} = \{1\}$ dela frogatzea. Horretarako, izan bitez $x,y \in N$ eta $H = \langle x,y\rangle \leq N$. Horrela, $H/Z(H)$ taldea 2 sortzaileko talde nilpotentea denez eta haren exponenteak m zatitzen duenez, finitua da eta $k = |H : Z(H)|$ zenbakiak $\eta(m)$ zatitzen du. Halaber, $\theta\colon H \to Z(H)$, $h \mapsto h^k$ funtzioa H-ren transferra da $Z(H)$-ra (konparatu [65, Teorema 10.1.3]ren frogarekin). Bereziki, θ homomorfismoa da eta $(xy)^k = x^k y^k$ da. Horrela x^k eta y^k elkarrekin trukatzen direnez eta k zenbaki osoak $\eta(m)$ zatitzen duenez,

$$(xy)^{\eta(m)} = x^{\eta(m)}y^{\eta(m)} \ \forall x,y \in N$$

da. Bereziki, $\theta'\colon N \to Z(N)$, $n \mapsto n^{\eta(m)}$ talde homomorfismoa da. Beraz, $\operatorname{im}\theta'$ abeldarra denez, $\gamma_2(N) \leq \ker\theta'$ da, hau da, $\gamma_2(N)^{\eta(m)} = \{1\}$. □

Datozen emaitzek talde aljebraiko linealen teoriako ideiak darabiltzate. Irakurleak [37]ra jo dezake emaitza horien atzeko teorian sakondu nahi badu.

Izan bedi K gorputz aljebraikoki itxia. Testu honetan zehar *talde aljebraiko lineal* bat $\mathrm{GL}_n(K)$-ren $\mathcal{G}$ azpitalde Zariski itxi bat izanen da, eta $\mathcal{G}$-ren *identitate osagaia* identitatearen osagai konexua da.

Proposizioa 4.17 (cf. [37, Proposizioa 7.3])**.** *Izan bedi $\mathcal{G}$ talde aljebraiko lineal konexua.*

(i) *Orduan, $\mathcal{G}^\circ$ indize finituko azpitalde normala da.*

(ii) *Izan bedi, $\mathcal{H} \leq \mathcal{G}$ indize finituko azpitalde itxi konexua, orduan $\mathcal{H} = \mathcal{G}^\circ$ da.*

†Zenbaki hau finitua da, Baerrek [4] frogatu zuenez, talde nilpotenteek Burnsideren problema betetzen dutelako (ikusi [20, Teorema 2.23]).

Matrize bat *unipotentea* da haren autobalio bakarra 1 bada, eta $\mathrm{GL}_n(K)$-ren azpitalde bat *azpitalde unipotentea* da elementu guztiak unipotenteak badira. Talde horien inguruko egiturazko emaitza nagusia talde unipotente oro $U_n(K)$-ren –diagonalean 1ak dituzten matrize goi triangeluarren taldea– azpitalde baten konjokatua dela da (ikusi [37, Korolarioa 17.5]). Bereziki, talde unipotente guztiak nilpotenteak dira, eta oinarriko K gorputza karakteristika positibokoa bada eta $\mathcal{G} \subseteq \mathrm{GL}_n(K)$ talde unipotentea, $\mathcal{G}$-k exponente finitua du, konparatu [38, Kapitulua III, Lema 16.2 eta Teorema 16.5] (nahiz eta erreferentzia gorputz finituetarako izan, argumentuek exponente positiboko gorputzetarako berdin-berdin balio dute).

Bestalde, $\mathcal{G}$ talde aljebraiko lineala emanda, haren *erradikal unipotentea*, $R_u(\mathcal{G})$ izendatuko duguna, $\mathcal{G}$-ko elementu unipotente guztiek osatutako azpitaldea da, edo baliokideki $\mathcal{G}$-ren azpitalde unipotente konexu handiena. Horrela, $R_u(\mathcal{G})$ konexua eta nilpotentea denez, $\mathcal{G}$-ren *erradikal ebazgarrian* dago, hau da, $\mathcal{G}$-ren azpitalde ebazgarri handienaren identitate osagaiaren barruan. Halaber, $\mathcal{G}$ talde aljebraiko lineala *erreduktiboa* dela diogu konexua bada eta $R_u(\mathcal{G})$ tribiala bada.

Eraikuntza horiek guztiak talde aljebraiko linealen teorian garrantzia handikoak dira, haatik soilik definizio nagusiak laburtu eta haien arteko erlazioak enuntziatuko ditugu. Izan ere, honako emaitza teknikoa baino ez dugu behar:

Proposizioa 4.18 (cf. [37, Lema 17.9])**.** *Izan bitez $\mathcal{G}$ talde aljebraiko lineal konexua eta $\mathcal{N}$ haren erradikal ebazgarria. Orduan, $[\mathcal{N},\mathcal{G}]$ unipotentea da.*

Froga. Izan bedi $\mathcal{R}_u(\mathcal{G})$ erradikal unipotentea. Orduan, $\mathcal{G}/\mathcal{R}_u(\mathcal{G})$ erreduktiboa da. Beraz, [37, Lema 17.9]ren arabera, ${}^{\mathcal{N}}/_{\mathcal{R}_u} \subseteq Z\left({}^{\mathcal{G}}/_{\mathcal{R}_u}\right)$ da eta, ondorioz, $[\mathcal{N},\mathcal{G}] \subseteq \mathcal{R}_u(\mathcal{G})$ da. □

Orain atal honetan bila genbiltzan emaitza froga dezakegu:

Teorema 4.19. *Talde $\mathbb{F}_p[[t]]$-analitiko trinkoak hitzez laburrak dira.*

Froga. Izan bitez G talde $\mathbb{F}_p[[t]]$-analitiko trinkoa eta w hitza. Demagun $w\{G\}$ finitua dela. Lehenik eta behin, Lema 4.15 dela eta, $w(G)'$ finitua da. Ondorioz, behar izanez gero zatidura batera pasata, orokortasunik galdu gabe $w(G)$ finituki sortutako talde abeldarra dela suposa dezakegu.

Korolarioa 4.13ren arabera, existitzen da S talde $\mathbb{F}_p[[t]]$-estandarra non w legea den. Laburtzearren izenda ditzagun $Z = Z(S)$ eta $K = \mathbb{F}_p((t))^{\mathrm{alg}}$, $\mathbb{F}_p((t))$ gorputz lokalaren itxitura aljebraikoa. Proposizioa 2.3gatik, S/Z lineala da $\mathbb{F}_p[[t]]$-ren gainean, eta, ondorioz, lineala da $\mathbb{F}_p((t))$ eta K gorputzen gainean ere. Gainera, Titsen alternatiba topologikoa (loc. cit.) dela eta, S/Z-k azpitalde ebazgarri ireki bat edo azpitalde aske dentso bat du barruan. Baina S/Z taldeak lege bat betetzen duenez, birtualki ebazgarria izan behar da.

Izan bedi $\mathcal{S}$ taldea S/Z-ren Zarizki itxitura $\mathrm{GL}_n(K)$-n, orudan $\mathcal{S}$ birtualki ebazgarria den Proposizioa 3.34ren arabera, eta izan bitez $\mathcal{N}$ $\mathcal{S}$-ren erradikal ebazgarria, hau da, azpitalde ebazgarri konexu handiena eta $\mathbb{N}^\circ$ bere osagai konexua, hau da, $\mathcal{S}$-ren erradikal ebazgarria. Proposizioa 4.17(i) dela eta, $\mathcal{N}$ indize finituko azpitaldea da $\mathcal{S}$-n, beraz, Proposizioa 4.17(ii)ren eraginez, $\mathcal{N}^\circ = \mathcal{S}^\circ$ da.

Izan bedi N/Z taldea S/Z-ren ebakidura $\mathcal{N}^\circ$-rekin, orduan, behar izanez gero muina normalera pasata, N indize finituko azpitalde normala da G-n. Proposizioa 4.18ren arabera, $[\mathcal{N}^\circ, \mathcal{N}^\circ]$ unipotentea da. Bereziki, $[\mathcal{N}^\circ, \mathcal{N}^\circ]$ nilpotentea da eta, K gorputzak karakteristika positiboa duenez, exponente finitua du. Hortaz, $[N,N]Z/Z$ exponente finituko talde nilpotentea da. Horrenbestez, $[N,N]Z$ nilpotentea da eta Lema 4.16ren arabera, $H := [N,N,S]$-k exponente finitua du.

Alde batetik,

$$H = [N,N,S] \geq [N,N,N]$$

da, eta, beraz, G/H birtualki gehienez 2 klaseko talde nilpotentea da. Horrela, $|w\{G/H\}| \leq |w\{G\}|$ denez eta G/H birtualki nilpotentea denez, $w\,(G/H)$ finitua da Turner-Smithen Teoremaren arabera (ikusi [72, Korolarioa 2]).

Beste alde batetik, $w(G) \cap H$ finituki sortutako talde abeldarra da eta exponente finitua du, beraz finitua da. Azkenik,

$$w\left(\frac{G}{H}\right) = \frac{w(G)H}{H} \cong \frac{w(G)}{w(G) \cap H}$$

isomorfismoak ematen du emaitza. □

Korolarioa 4.20. *Izan bedi R bat Krullen dimentsioko pro-p domeinua. Orduan, talde R-analitiko trinkoak hitzez laburrak dira.*

4.3 Laburtasuna talde R-analitiko trinkoetan

Prest gaude emaitza nagusia frogatzeko.

Teorema 4.21. *Talde R-analitiko trinkoak hitzez laburrak dira.*

Froga. Korolarioa 4.20 aintzat hartuta, demagun R pro-p domeinuak gutxienez 2 Krull dimentsioa duela. Izan bitez G talde R-analitiko trinkoa eta w hitz bat k aldagaitan non $w\{G\}$ finitua den. Lema 4.15 dela eta, nahikoa da w-balio guztiek ordena finitua dutela frogatzea.

Lema 1.23ren arabera, existitzen da (S, ϕ) azpitalde R-estandar ireki normala non $g \in G$ guztietarako $c_g \colon S \to S$, $x \mapsto x^g$ konjokazio aplikazioak hertsiki analitikoak diren. Horrela, $n = |G : S|$ bada, $w^n\{G\} \subseteq S$ da eta, Lema 4.15en arabera, $w^n(G)$ finitua da baldin eta soilik baldin $w(G)$ is finitua bada. Hortaz, orokortasunik galdu gabe suposa dezagun $w\{G\} \subseteq S$ dela.

Izan bedi $(P, \mathfrak{m})$ ideal nagusietako pro-p domeinua hau: $P = \mathbb{Z}_p$, $\operatorname{char} R = 0$ denean, eta $P = \mathbb{F}_p[[t]]$, $\operatorname{char} R = p$ positiboa denean. Cohenen Egitura Teoremaren arabera (ikusi Teorema 1.2), R eraztuna $P[[t_1, \ldots, t_m]]$-ren finituki sortutako eraztun hedadura integrala da, $m = \dim_{\mathrm{Krull}}(R) - 1$ izanik. Horrela, $a \in \mathfrak{m}^{(m)}$ bakoitzerako, izan bedi $s_a \colon P[[t_1, \ldots, t_m]] \to P$, $F(t_1, \ldots, t_m) \mapsto F(a)$ ebaluazio homomorfismoa. Korolarioa 2.13k s_a epimorfismoa $\tilde{s}_a \colon R \to Q$ eraztun epimorfismora hedatzen du, non Oharra 2.14ren arabera, $Q = (Q, \mathfrak{n})$ pro-p domeinua P-ren finituki sortutako eraztun hedadura integrala den, bereziki Q-ren Krullen dimentsioa 1 da.

Izan bedi $a \in \mathfrak{m}^{(m)}$, Atala 2.2ko notazioa jarraituz, froga honetan zehar $\mathbf{W}_a$ moduan izendatuko dugu $\mathbf{W}_{\tilde{s}_a} \in Q[[\mathbf{X}]]^{(l)}$ berretura serie formalen tupla, zeinnahi $\mathbf{W} \in R[[\mathbf{X}]]^{(l)}$ berretura serie formalen tuplatarako. Bereziki, $\mathbf{F}$ bada S-ren talde eragiketa formala, orduan $\mathbf{F}_a = \mathbf{F}_{\tilde{s}_a}$ talde eragiketa formala da (ikusi Korolarioa 2.8). Izan bedi T ezker transbertsal bat S-rentzat G-n, eta suposa dezagun $1 \in T$ dela. Hemendik aurrera S-tik eratorritako atlasaz baliatuko gara, hots, $\{(tS, \phi_t)\}_{t \in T}$ non $\phi_t(x) := \phi\left(t^{-1}x\right)$ den (konparatu Atala 1.3).

Lema 2.10 erabilita, definitu $L := \left(\mathfrak{n}^N\right)^{(d)}$ talde Q-estandarra, zeinaren talde eragiketa $\mathbf{F}_a$ talde eragiketa formalak definitzen duen, eta $H := T \times L$ talde Q-analitikoa (2.3)ko eragiketarekin, zeina L-ren gaintaldea balitz bezala ikus daitekeen. Oroitu H-ren egitura Q-analitikoa $\{(tL, \psi_t)\}_{t \in T}$, non $\psi_t(t, l) = l$, atlasak ematen duela.

Atal hau bukatu arte finka dezagun $(t_1, \ldots, t_k) \in T^{(k)}$ tupla, eta demagun, (4.2) dela eta, edozein $l \in \mathbb{N}$ zenbakitarako w^l hitz funtzioa $t_1S \times \cdots \times t_kS$ multzo irekian $\mathbf{W}^l$ berretura serie formalen tuplak emanda dagoela, hau da, $w\{G\} \subseteq S$ dela aintzat hartuta,

$$\phi\left(w^l(x_1, \ldots, x_k)\right) = \mathbf{W}^l\left(\phi_{t_1}(x_1), \ldots, \phi_{t_k}(x_k)\right) \ \forall x_j \in t_jS$$

dugu (notazioa arintzearren explizituki idatziko ez den arren, kontuan eduki $\mathbf{W}^l$ berretura seriea $t_1, \ldots, t_k$ balioen menpekoa ere badela).

Izan bedi $w^l \colon H^{(k)} \to H$ hitz funtzioa $*_a$ eragiketarekin H-n. Lema 2.7 eta Oharra 2.11 direla eta,

$$\psi_1\left(w^l(x_1, \ldots, x_k)\right) = \mathbf{W}_a^l\left(\psi_{t_1}(x_1), \ldots, \psi_{t_k}(x_k)\right) \ \forall x_j \in t_jL$$

da.

Gainera, Proposizioa 4.14ren arabera, $w\{G\}$ finitua denez, S azpitalde marjinala da w-rentzat, hau da, w funtzioa konstantea da $t_1S\times\cdots\times t_kS$ multzo irekian. Hortaz, $w\colon H^{(k)}\to L$ hitz funtzioa konstantea da $t_1'L\times\cdots\times t_k'L$ multzo bakoitzean ($t_j'\in T$), eta $|w\{H\}|\leq |H:L|^k=n^k$ finitua da. Korolarioa 4.20ren arabera, H hitzez laburra da, eta, beraz, existitzen da $\ell_a\in\mathbb{N}$ non $w^{\ell_a}(H)=\{(1,\mathbf{0})\}$ den. Bereziki,

$$\mathbf{W}_a^{\ell_a}(X_1,\ldots,X_{dk})=\mathbf{0} \tag{4.5}$$

da. Definitu $\mathfrak{m}^{(m)}$ espazioaren partiketa hau:

$$\mathfrak{m}_\ell=\left\{a\in\mathfrak{m}^{(m)}\middle|\mathbf{W}_a^\ell=\mathbf{0}\right\},\ \ell\in\mathbb{N}.$$

Horrela, $\mathfrak{m}^{(m)}=\bigcup_{\ell\in\mathbb{N}}\overline{\mathfrak{m}_\ell}$ eta $\mathfrak{m}^{(m)}$ osoa direnez, Baireren Kategoria Teoremaren arabera, existitzen da ℓ' zenbaki osoa non $\overline{\mathfrak{m}_{\ell'}}$ multzoak $V\subseteq_o\mathfrak{m}^{(m)}$ azpimultzo ireki ez-hutsa barruan duen. Halaber, $\mathbf{W}^{\ell'}$ berretura serie formalen tupla konstantea da, hau da,

$$\mathbf{W}^{\ell'}(X_1,\ldots,X_{dk})=(c_1,\ldots,c_d)\in R^{(d)}$$

da, eta $a\in\mathfrak{m}_{\ell'}$ guztietarako

$$(\tilde{s}_a(c_1),\ldots,\tilde{s}_a(c_d))=\mathbf{W}_a^{\ell'}(X_1,\ldots,X_{dk})=\mathbf{0}$$

betetzen da. Hots, $\tilde{s_a}(c_i)=0$ da $a\in\mathfrak{m}_{\ell'}\cap V$ guztietarako. Bestetik, $\mathfrak{m}_{\ell'}\cap V$ dentsoa da V-n, eta Korolarioa 2.15 dela eta, $c_i=0$ da $i\in\{1,\ldots,d\}$ guztietarako. Azkenik, prozedura hau $T^{(k)}$-ko tupla guztietarako errepikatuz, ℓ zenbaki osoa erdiesten dugu non $\phi\left(w^\ell(x_1,\ldots,x_k)\right)=\mathbf{0}$ den $x_i\in G$ guztietarako, alegia $w^\ell(G)=\{1\}$ da. □

Jakina da pro-p talde uniformeki berreturabeteak tortsiogabeak direla (ikusi [24, Teorema 4.5]). Hori dela eta, H talde p-adiko analitiko trinkoa bada, $L\leq H$ azpitalde $\mathbb{Z}_p$-estandar (eta, beraz, uniformeki berreturabete) ireki bat du. Areago, w hitzerako $w\{H\}$ finitua bada, orduan $w(H)$ finitua da. Horrenbestez, $w^{|H:L|}\{H\}$-ko elementuek ordena finitua dute eta aldi berean L talde tortsiogabean daude. Hots, $w^{|H:L|}$ legea da H-n.

Aurrekoa aintzat hartuta, emaitza nagusiaren froga apur bat sinplifika zitekeen R zero karakteristikako pro-p domeinua denean. Izan ere, (4.5) identitatean $\mathbf{W}_a^{|G:S|}=\mathbf{0}$ izanen genuke, zein-nahi $\tilde{s}_a$ ebaluaziotarako, eta, beraz, Baireren Kategoria Teorema erabiltzea ekidin genezakeen. Ohartu, bide batez, horrek frogatuko lukeela w-balioen ordenak $|G:S|$ zatitzen duela.

Talde profinituetan laburtasun nozioa indartu daiteke eta laburtasun gogorra definitu (ikusi [22] eta [23]). Zehazkiago, w hitza $\mathcal{C}$ talde klasean *gogorki laburra* dela diogu $G \in \mathcal{C}$ guztietarako $|w\{G\}| < 2^{\aleph_0}$ izateak $w(G)$ hitzezko azpitaldea finitua dela inplikatzen badu.

Halere, talde R-analitiko trinkoetan laburtasuna eta laburtasun gogorra baliokideak dira.

Lema 4.22 (cf. [22, Proposizioa 2.1])**.** *Izan bedi X espazio topologiko trinkoa eta Y* espazio profinitua*, hau da, espazio topologiko Hausdorff trinko guztiz diskonexua. Demagun $F\colon X \to Y$ funtzio jarraitua ez dela inon konstantea dela, hau da, $U \subseteq_o X$ ireki guztietarako $F|_U$ ez da konstantea. Orduan, $|\operatorname{im} F| \geq 2^{\aleph_0}$ da.*

Froga. Izan bedi $U \subseteq_o X$ multzo itxia eta irekia (e.g. $U = X$). Orduan, $F|_U$ ez denez konstantea, badaude bi azpimultzo $U_1, U_2 \subseteq U$, aldi berean itxiak eta irekiak, non $F(U_1) \cap F(U_2) = \varnothing$ den. Prozedura hau errepikatuz, $\mathbf{x} = (x_n)_{n\geq 1} \in \{1,2\}^{\mathbb{N}}$ segidarako existititzen da

$$U \supseteq U_{x_1} \supseteq U_{x_1,x_2} \supseteq \cdots \supseteq U_{x_1,\ldots,x_n} \supseteq \ldots$$

azpimultzo itxien segida beherakorra. Definitu $U_{\mathbf{x}} := \bigcap_{n\in\mathbb{N}} U_{x_1,\ldots,x_n}$, zeina ezhutsa den, X trinkoa baita. Horrela, $\mathbf{x} \in \{1,2\}^{\mathbb{N}}$ segida bakoitzerako, hautatu $u_{\mathbf{x}} \in U_{\mathbf{x}}$ elementua. Orduan, $F(u_{\mathbf{x}}) \neq F(u_{\mathbf{y}})$ denez, F funtzioa

$$\mathcal{S} := \left\{u_{\mathbf{x}} \mid \mathbf{x} \in \{1,2\}^{\mathbb{N}}\right\}$$

multzoan injektiboa da, eta, beraz, $|F(X)| \geq |F(\mathcal{S})| = 2^{\aleph_0}$ da. □

Proposizioa 4.23. *Izan bitez M eta N barietate R-analitiko trinkoak eta $F\colon M \to N$ funtzio R-analitikoa. Demagun, $|\operatorname{im} F| < 2^{\aleph_0}$ dela, orduan* $\operatorname{im} F$ *finitua da.*

Froga. Izan bitez $m = \dim_x M$ eta $n = \dim_{F(x)} N$. Orduan, F analitikoa denez, $x \in M$ guztietarako existitzen da x puntuaren (U_x, ϕ, m) R-karta erregularra M-n, $F(x)$ puntuaren (V_x, ψ, n) R-karta N-n eta $\mathbf{G} \in R[[X_1, \ldots, X_m]]^{(n)}$ berretura serieen tupla non

$$\psi \circ F \circ \phi^{-1}(y) = \mathbf{G}(y) \ \forall y \in \phi(U_x)$$

den. Horrela, $\phi(U_x) = z + \left(\mathfrak{m}^L\right)^{(m)}$ da $z \in R^{(m)}$ eta $L \in \mathbb{N}$ batzuetarako, bereziki U_x espazio profinitua. Hortaz, $F|_{U_x}\colon U_x \to N$ espazio profinituen arteko funtzio jarraitua da, eta hipotesiz $|F(U_x)| < 2^{\aleph_0}$ betetzen denez, Lema 4.22ren arabera, existitzen da $V \subseteq_o U_x$ irekia non $F|_V$ konstantea den. Horregatik, Lema 1.8 dela eta, $F|_{U_x}$ konstantea da. Trinkotasunagatik $M = \bigcup_{z\in Z} U_z$ da $Z \subseteq M$ azpimultzo finitu baterako, eta, ondorioz, $|\operatorname{im} F| \leq |Z|$ da. □

Bereziki, talde R-analitiko trinko batean $|w\{G\}| < 2^{\aleph_0}$ bada, $w\{G\}$ finitua da. Hori dela eta, kapitulu honetako emaitza nagusia honela berridatz daiteke:

Korolarioa 4.24. *Hitz guztiak gogorki laburrak dira talde R-analitiko trinkoen klasean.*

Orobat, bide batez, zenbait finituki sortutako talde R-analitikotan hitz balioen multzoaren tamainarentzat behe-borne bat erdietsi dugu. Konkretuago, [45, Teorema 1.3]n, autoreek lege bat betetzen duten finituki sortutako talde R-analitiko trinkoak bakartu zituzten, talde p-adiko analitikoak direla frogatu baitzuten.

Korolarioa 4.25. *Izan bitez R pro-p domeinua eta G finituki sortutako talde R-analitiko ez-diskretua. Demagun R-ren Krullen dimentsioa gutxienez 2 dela edo R-ren karakteristika positiboa dela. Orduan, w hitz guztietarako $|w\{G\}| \geq 2^{\aleph_0}$ da.*

Froga. Lema 1.21en arabera, existitzen da $S \leq_o G$ azpitalde R-estandar irekia. Demagun $|w\{G\}| < 2^{\aleph_0}$ dela, orduan $|w\{S\}| < 2^{\aleph_0}$ da, eta S trinkoa denez, Proposizioa 4.12 eta Proposizioa 4.23 direla eta, w legea da S-n. Beraz, [45, Teorema 1.3]ren arabera, S-k aldi berean barietate egitura p-adiko analitikoa eta R-analitikoa onartzen ditu, eta hori kontraesana da Teorema 1.45 (i)ekin. □

4.5 Oharrak

Hitzen eta hitz zabaleraren inguruko material gehigarria Neumann [61] eta Segalen [67] liburuetan topa daiteke. Halaber, P. Hall [33] eta Turner-Smithen [72] jatorrizko artikuluak irakurtzeak ere merezi du, oinarrizko nozioak bertan zehazten baitira. Gainera, laburtasun gogor kontzeptua talde profinituen testuinguruan [22] eta [23] artikuletan ageri da lehenbizikoz. Azkenik, “hitz sendo” terminoa [69]tik hartu da.

Kapitulu honen ekarpen nagusia Teorema 4.21 da, [76]en argitaratua. Halere, Teorema 4.19 hori frogatzeko urrats garrantzitsua da, eta Jaikin-Zapirain eta Klospchen [45] lanean oinarritu gara horretarako.

A

Adoren Teorema ideal nagusietako domeinuetan

Atala 2.1en, Adoren Teorema ospetsuaren aldaera bat behar genuen: Lie aljebra ideal nagusietako domeinu baten gainean definituta dago, eta, aldi berean, adierazpen matrizialaren maila soilik Lie aljebraren heinaren, R-modulu aske gisa, menpekoa izan behar da. Lehenengoa jakina da, [19] edota [74]n frogatzen da eta, baina ez dugu bigarrenerako erreferentzia zehatzik aurkitu. Horregatik, tesiaren osotasuna bermatzeko, apendize honetan teoremaren beharrezko bertsioa frogatuko da, literatura matematikoan gai honen inguruan ageri diren hainbat ideia bilduz.

A.1 Sarrera

Apendize honetan zehar R zero karakteristikako ideal nagusietako domeinu (IND) bat eta K haren zatikien gorputza izanen dira. Halaber, R-aljebra diogunean, besterik zehaztu ezean, ez du zertan identitateduna, elkarkorra edo trukakorra izan.

Gogoratu *R-Lie aljebra* bat, deitu $\mathfrak{L}$, R-modulu bat dela, $[\cdot,\cdot]\colon \mathfrak{L}\times\mathfrak{L}\to\mathfrak{L}$ *Lieren kortxete* batekin batera, hau da, aplikazio R-bilineal antisimetriko bat zeinak *Jacobiren identitatea* betetzen duen:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0,\ \forall x,y,z \in \mathfrak{L}.$$

Lie aljebra homomorfismoak kortxeteak mantentzen dituzten aplikazio linealak dira, hau da, $\phi\colon \mathfrak{L} \to \mathfrak{L}'$ aplikazio linealak non

$$\phi\left([x,y]_{\mathfrak{L}}\right) = [\phi(x),\phi(y)]_{\mathfrak{L}'}\ \forall x,y \in \mathfrak{L}.$$

Aljebra horien adibide nagusia *Lie aljebra konkretuak* dira, hau da, W R-modulu askearen gaineko R-modulu endomorfismoen $\mathrm{End}_R(W)$ taldea

$$[f,g] = f \circ g - g \circ f$$

kortxeatearekin hornituta.

Definizioa A.1. Izan bedi $\mathfrak{L}$ Lie aljebra R-ren gainean. Orduan, $\mathfrak{L}$-ren *Lie aljebra adierazpen* bat

$$\phi\colon \mathfrak{L} \to \mathrm{End}_R(W)$$

R-modulu homomorfismo bat da, W R-modulu aske batentzat, eta ϕ *adierazpen leiala* da ϕ funtzioa injektiboa denean.

Halaber, ϕ-ren *maila*, $\deg\phi$ adieraziko dena, W-ren heina da, eta ϕ *maila finituko adierazpena* edo *adierazpen finitua* dela esanen dugu $\deg\phi$ finitua denean. Baliokideki, $\mathrm{End}_R(W) \cong \mathrm{M}_{\mathrm{rk}\,W}(R)$ denez Lie aljebra gisa (matrizeak $[A,B] = AB - BA$ Lieren kortxetearekin hornituta), r mailako adierazpen leialei r mailako *adierazpen matrizial* ere baderitze.

Gorputzen gainean Lie aljebrak asko aztertu dira, eta Adoren Teoremak horien adierazgarritasunaren berri ematen du:

Teorema A.2 (Ado [2])**.** *Demagun R zero karakteristikako gorputza dela. Izan bedi $\mathfrak{L}$ dimentsio finituko R-Lie aljebra. Orduan, $\mathfrak{L}$-k adierazpen finitu leial bat du.*

Orokorrago, $\mathfrak{L}$-ri R-modulu askea izatea eskaten badiogu, aurreko teorema IND-etara heda daiteke:

Teorema A.3 (Churkin [19]; Weigel [74])**.** *Izan bitez R zero karakteristikako INDa eta $\mathfrak{L}$ heina finituko R-modulu askea den R-Lie aljebra. Orduan, $\mathfrak{L}$-k adierazpen finitu leial bat du.*

Hemendik aurrera [74]ko notazioa jarraituz, *R-Lie erretikulu* bat heina finituko R-modulu askea den R-Lie aljebra bat da. Gorputzen gainean Adoren Teoremaren froga anitz daude (ikusi, esate baterako, [41, Kapitulua VI, 2. Atala]), eta horietako gehienetatik ondorioztatu daiteke eraikitzen den adierazpenaren maila soilik $\dim_K \mathfrak{L}$, aljebraren K-espazio bektorial dimentsioaren, menpekoa dela. Zehatzago esanda, izan bedi

$$\deg \mathfrak{L} := \min\{\deg \phi \mid \phi \ \mathfrak{L}\text{-ren adierazpen leiala da}\},$$

orduan [13] eta [60]ko argudioetan oinarrituta, ikus daiteke R zero karakteristikako gorputza eta $r = \dim_K \mathfrak{L}$ direnean:

$$\deg \mathfrak{L} \leq \alpha \frac{2^r}{\sqrt{r}} \tag{A.1}$$

dela, $\alpha \in \mathbb{R}$ batentzat (ikusi [58, 1.1.2 Atala] froga zehatz baterako). Alabaina, badaude gorputzen gainean $\deg \mathfrak{L}$ aztertzen duten beste hainbat lan ere; aipatu beharrekoak dira [8], [31] edota [63] lanak.

Haatik, R IND orokor bat denean, aipatutako bi frogetatik ez da zuzenean ondorioztatzen $\deg \mathfrak{L}$ zenbaki osoa soilik $\operatorname{rk} \mathfrak{L}$, erretikuluaren heinaren, menpekoa denik. Areago, [74, Proposizioa 3.4]n gerora adierazpenaren maila izanen dena finitua da R eraztun noetherdarra izateagatik ideal segida bat geldikorra delako, baina ezin da zehaztu zenbat idealek osatzen duten segida hori.

Apendize honetan, R-Lie erretikulu baten adierazpen leial bat eraikitzeko modu kuantitatibo bat aurkeztuko dugu, [8] eta [63]ko ideietan oinarrituz. Zehatzago:

Teorema A.4. *Izan bitez R zero karakteristikako INDa eta $\mathfrak{L}$ r heinako R-Lie erretikulua. Orduan,*

$$\deg \mathfrak{L} \leq r + \sqrt{\frac{r+1}{r}4^r}$$

da.

Orobat, aipatu karakteristika positiboko gorputzetarako Adoren Teoremaren parekoa betetzen dela, hori da Iwasawaren Teorema [40] hain zuzen ere. Are gehiago, Teorema A.2ren bertsio orokorrari, koefizienteen gorputzari beste inolako baldintzarik ezarri gabe, Ado-Iwasawaren Teoerema deitzen zaio. Karakteristika positiboan emaitza orokortasun askoz gehiagorekin eman daiteke:

Teorema A.5 (cf. [19, Teorema 3])**.** *Izan bitez R karakteristika positiboko eraztun trukakorra eta $\mathfrak{L}$ heina finituko R-Lie erretikulua. Orduan, existitzen dira W heina finituko R-modulu askea eta $\phi\colon \mathfrak{L} \to \operatorname{End}_R(W)$ R-Lie aljebra monomorfismoa.*

Hori frogatzeko nahikoa da froga originala hitzez hitz errepikatzea, eta, horrenbestez, gorputzen gainean lortzen den borne bera lortzen da R-Lie erretikuluen mailarentzat. Hots,

$$\deg \mathfrak{L} \leq n^{\operatorname{rk}^3 \mathfrak{L}},$$

$n = \operatorname{char} R$ izanik (ikusi [5, Atala 6.2.4]).

Oharrak. Frogetan zehar R-modulu askeen (R INDa izanik) inguruko zenbait propietate erabiliko dira. Honatx gogoratu beharrekoak:

(i) M R-modulu aske baten azpimoduluak askeak dira, eta gehienez $\operatorname{rk}(M)$ heina dute.

Izan bitez M R-modulua eta $N \leq M$ azpimodulua. N-ren *isolatzailea* M-n

$$\operatorname{Iso}_M(N) = \{x \in M \mid \exists r \in R \setminus \{0\} \text{ non } rx \in N\}$$

azpimodulua da, eta N M-n *isolatua* dela diogu $\operatorname{Iso}_M(N) = N$ denean.

(ii) $M/\operatorname{Iso}(N)$ tortsiorik gabeko R-modulua da.

(iii) M R-modulu askea, $N \leq M$ azpimodulu isolatua eta M/N R-modulu finituki sortua badira, M/N R-modulu askea da, eta

$$\operatorname{rk}(M) = \operatorname{rk}(N) + \operatorname{rk}(M/N)$$

da. Kasu honetan, $\operatorname{rk}(M/N)$ zenbakiari N-ren *koheina* deritzogu.

(iv) M heina finituko R-modulu askea eta N azpimodulu isolatua badira, N-k osagarria dauka M-n, hau da, existitzen da L R-modulu askea non $M = N \oplus L$ den.

A.2 Adierazpen adjuntua eta erregularra

Aurkez ditzagun zein-nahi Lie aljebraren bi adierazpen. Alde batetik, $x \in \mathfrak{L}$ elementuak $\operatorname{ad}_x \colon \mathfrak{L} \to \mathfrak{L}$, $y \mapsto [x, y]$ aplikazio lineala definitzen du. Jacobiren identitatea dela eta, esleipen horrek $\mathfrak{L}$-ren $\operatorname{rk} \mathfrak{L}$ mailako *adierazpen adjuntua* definitzen du, hau da,

$$\operatorname{Ad} \colon \mathfrak{L} \to \operatorname{End}_R(\mathfrak{L}), \; x \mapsto \operatorname{ad}_x .$$

Halere, adierazpen hori orohar ez da leila, haren nukleoa $\mathfrak{L}$-ren *zentroa* baita, hau da,

$$Z(\mathfrak{L}) := \{x \in \mathfrak{L} \mid [x, y] = 0 \; \forall y \in \mathfrak{L}\} .$$

Horrenbestez, $\mathfrak{L}$ R-aljebra *semisinplea* bada –hots, ez badu ideal abeldar ez-tribialik–, adierazpen adjuntua leiala da eta $\deg \mathfrak{L} \leq \operatorname{rk} \mathfrak{L}$ da.

Bigarren adierazpena aurkezteko inguratze aljebra unibertsala definitu behar da.

Definizioa A.6 (cf. [41, Kapitulua V, Teorema 1.1])**.** Izan bedi $\mathfrak{L}$ R-Lie aljebra. $\mathfrak{L}$-ren *R-tentsore aljebra*

$$\mathbf{T}_R(\mathfrak{L}) = R \oplus \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \cdots \oplus \mathfrak{L}_i \oplus \dots$$

da, $\mathfrak{L}_i = \mathfrak{L} \otimes \overset{(i)}{\dots} \otimes \mathfrak{L}$ tentsio R-modulua delarik. Horren R-modulu egitura biderketa tentsorialarena da, eta biderketa

$$(x_1 \otimes \cdots \otimes x_i) \otimes (y_1 \otimes \cdots \otimes y_i) = x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_i$$

arauak definitzen du. Izan bedi $\mathfrak{R}$

$$[u, v] - (u \otimes v - v \otimes u), \; u, v \in \mathfrak{L}$$

elementuek sortutako $\mathbf{T}_R(\mathfrak{L})$-ren ideala. Orduan, $\mathfrak{L}$-ren *inguratze aljebra unibertsala*

$$\mathcal{U}_R(\mathfrak{L}) := \frac{\mathbf{T}_R(\mathfrak{L})}{\mathfrak{R}}$$

identitatedun R-aljebra elkarkorra da.

Gainera, $\mathfrak{L}$ eta $\mathfrak{L}_1$ elkarrekin identifikatuz gero, $\iota \colon \mathfrak{L} \to \mathcal{U}_R(\mathfrak{L})$ homomorfismoa dugu. Ikus daiteke $\mathfrak{L}$ finituki sortua eta R INDa direnean, ι injektiboa dela (ikusi [74, Teorema 3.2]), eta, beraz, $\mathfrak{L} \subseteq \mathcal{U}_R(\mathfrak{L})$ dela asumi dezakegu. Halaber, notazioa sinplifikatzearren, $x_1 \otimes \cdots \otimes x_k$ elementua $x_1 \dots x_k$ monomioa bezala idatziko dugu. Aljebra unibertsala Poincaré-Birkhoff-Witten teoremak deskribatzen du:

Teorema A.7 (cf. [74, Teorema 3.2])**.** *Izan bitez $\mathfrak{L}$ r heinako R-Lie erretikulua eta $\{x_1, \dots, x_r\}$ haren oinarria. Orduan, $\mathcal{U}_R(\mathfrak{L})$ R-modulu askea da, eta*

$$\{x_1^{\alpha_1} \dots x_r^{\alpha_r} \mid \alpha_i \in \mathbb{N}_0\} \tag{A.2}$$

monomioek oinarri bat osatzen dute.

Bi monomio emanda, haien biderketa (A.2) moduko monomioen konbinazio lineal gisa adieraz daiteke, hurrenez hurren $x_j x_i = x_i x_j - [x_i, x_j]$ identitatea erabiliz indeterminatuak ordenatzeko.

Edozein $\mathfrak{L}$ R-Lie erretikuluk $\mathcal{U}_R(\mathfrak{L})$-ren gainean eragiten du ezker biderketaz. Hots, $x \in \mathfrak{L}$ bakoitzerako $\ell_x\colon \mathcal{U}_R(\mathfrak{L}) \to \mathcal{U}_R(\mathfrak{L})$, $u \mapsto xu$ R-endomorfismoa dugu. Alde batetik, edozein $x, y \in \mathfrak{L}$-tarako

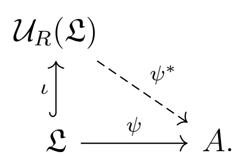
$$[x, y] = xy - yx$$

da $\mathcal{U}_R(\mathfrak{L})$-n eta, beraz, $\mathcal{L}\colon \mathfrak{L} \to \mathrm{End}_R(\mathcal{U}_R(\mathfrak{L}))$, $x \mapsto \ell_x$ R-Lie aljebra homomorfismoa da. Halaber, inguratze aljebra unibertsala identitateduna denez $x \neq y$ denean

$$\ell_x(1) = x \neq y = \ell_y(1)$$

da. Ondorioz, $\mathcal{L}$ adierazpen leiala da, (ezker) *adierazpen erregular* deituko duguna. Alta, $\mathcal{U}_R(\mathfrak{L})$ heina infinitukoa denez, adierazpen hori ez da finitua. Baina inguratze aljebra unibertsala karakterizatzen duen propietate hau dela eta, Lie aljebra adierazpen guztiek $\mathcal{U}_R(\mathfrak{L})$-ren zatidura baten gainean ekiten dute.

Teorema A.8 (Propietate unibertsala, cf. [11, Kapitulua I, § 2.1, Proposizioa 1])**.** *Izan bitez $\mathfrak{L}$ R-Lie erretikulua, A R-aljebra elkarkorra, $[a, b] = ab - ba$ Lieren kortxea ($a, b \in A$ guztietarako) eta $\psi\colon \mathfrak{L} \to (A, [,])$ R-Lie aljebra homomorfismoa. Orduan, existitzen da $\psi^*\colon \mathcal{U}_R(\mathfrak{L}) \to A$ R-aljebra homomorfismo bakarra non $\psi = \psi^* \circ \iota$ den. Alegia, diagrama hau trukakorra da:*



Hain zuzen ere, propietate unibertsal hori da $\mathcal{U}_R(\mathfrak{L})$-ren izenaren arrazoia. Are gehiago, froga daiteke propietate unibertsala betetzen duen edozein R-aljebra $\mathcal{U}_R(\mathfrak{L})$-ri isomorfoa dela (ikusi [41, Kapitulua V, Teorema 1.1.1]).

A.3 Adoren Teorema

Adoren Teorema frogatzeko askotan $\mathfrak{L}$ R-Lie erretikulutik $\mathfrak{L}_K := \mathfrak{L} \otimes_R K$ K-espazio bektorialera pasako gara. Ohartu $\mathfrak{L}_K$ K-aljebra bat dela eta haren Lieren kortxetea $\mathfrak{L}$-ren Lieren kortxeteak tentsorizatuz induzituriko K-aplikazio lineala dela. Kontuan hartu berehalako propietate hauek:

- $\mathfrak{L}_K$ espazio bektorialak $\mathrm{rk}\,\mathfrak{L}$ dimentsioa du,

- $\mathfrak{L} = \langle x_1, \dots, x_r \rangle_R$ bada, orduan $\mathfrak{L}_K = \langle x_1, \dots, x_r \rangle_K$ da eta
- $\mathfrak{I} \trianglelefteq \mathfrak{L}_K$ ideal guztietarako, $\mathfrak{I} \cap \mathfrak{L} \trianglelefteq \mathfrak{L}$ ideal isolatua da.

Teorema hiru urratsetan frogatuko dugu.

A.3.1 Lieren erretikulu nilpotenteak

Lehenik eta behin, demagun $\mathfrak{L}$ r heinako R-Lie erretikulu nilpotentea dela. Gogoratu, $\mathfrak{L}$ Lie erretikuluaren *serie zentral beherakorra* errekurtsiboki

$$\gamma_1(\mathfrak{L}) = \mathfrak{L},\ \gamma_i(\mathfrak{L}) = [\gamma_{i-1}(\mathfrak{L}), \mathfrak{L}]\ \forall i \geq 2$$

moduan definitzen dela, eta $\mathfrak{L}$ nilpotentea dela existitzen bada $\gamma_{c+1}(\mathfrak{L}) = \{0\}$ betetzen duen $c \in \mathbb{N}$ zenbaki osoa. Zenbaki oso horietan txikienari, existituz gero, R-Lie erretikuluaren *nilpotentzia klasea* deritzo. Biderketa tentsoriala lineala denez, erraz ikus daiteke honako emaitza hau:

Lema A.9. *Izan bitez $\mathfrak{L}$ R-Lie erretikulua eta $\mathfrak{I}, \mathfrak{H} \trianglelefteq \mathfrak{L}$ idealak. Orduan,*

$$[\mathfrak{I} \otimes_R K, \mathfrak{H} \otimes_R K] = [\mathfrak{I}, \mathfrak{H}] \otimes_R K$$

da. Beraz, $\mathfrak{L}$ c klaseko R-Lie erretikulu nilpotentea bada, $\mathfrak{L}_K$ c klaseko K-Lie aljebra nilpotentea da.

Orduan,

$$\gamma_0(\mathfrak{L}_K) > \gamma_1(\mathfrak{L}_K) > \cdots > \gamma_i(\mathfrak{L}_K) > \cdots > \gamma_{c+1}(\mathfrak{L}_K) = \{0\}$$

K-espazio bektorialen segida hertsiki beherakorra denez, $c \leq \dim \mathfrak{L}_K = r$ da. Definitu $\mathfrak{L}_i = \gamma_i(\mathfrak{L}_K) \cap \mathfrak{L} \trianglelefteq \mathfrak{L}$ ideal isolatuak, eta hartu $\mathfrak{L}$-ren $\{x_1, \dots, x_r\}$ oinarria non lehenengo $x_1, \dots, x_{r_1}$ elementuak $\mathfrak{L}_c$-ren oinarria diren, lehenengo $x_1, \dots, x_{r_2}$ ($r_2 > r_1$) elementuak $\mathfrak{L}_{c-1}$-en oinarria diren, eta horrela hurrenez hurren. Teorema A.7ren arabera,

$$\mathbf{x}^\alpha := x_1^{\alpha_1} \dots x_r^{\alpha_r},\ \alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}_0^{(r)}$$

monomioek $\mathcal{U}_R(\mathfrak{L})$ inguratze aljebra unibertsalaren oinarria osatzen dute. Hori kontuan hartuta defini dezagun $\omega\colon \mathcal{U}_R(\mathfrak{L}) \to \mathbb{N}_0 \cup \{\infty\}$ *pisu funtzioa* ondoko moduan:

$$\omega(x_i) = \max\{m \mid x_i \in \mathfrak{L}_m\} \qquad\qquad \omega(\mathbf{x}^\alpha) = \textstyle\sum_{i=1}^r \alpha_i \omega(x_i)$$

$$\omega\left(\textstyle\sum_{i=1}^r c_\alpha \mathbf{x}^\alpha\right) = \min\left\{\omega(\mathbf{x}^\alpha) \mid c_\alpha \neq 0\right\} \qquad \text{eta } \omega(0) = \infty.$$

Edozein $m \in \mathbb{N}_0$-tarako definitu

$$\mathfrak{U}^m(\mathfrak{L}) := \{u \in \mathcal{U}_R(\mathfrak{L}) \mid \omega(u) > m\}$$

–$\mathfrak{L}$ erretikulua zein den garbi dagoenean, $\mathfrak{U}^m$ baino ez dugu idatziko–. Ikus dezagun $\mathfrak{U}^m \trianglelefteq \mathcal{U}_R(\mathfrak{L})$ ideal isolatua dela:

(i) Ohartu $0 \in \mathfrak{U}^m$ dela m guztietarako, eta

$$\omega(rx) = \omega(x) \text{ eta } \omega(x+y) \geq \min\{w(x), w(y)\}$$

direla $x, y \in \mathcal{U}_R(\mathfrak{L})$ eta $r \in R \setminus \{0\}$ guztietarako. Gainera, $\omega([x_i, x_j]) \geq \omega(x_i) + \omega(x_j)$ denez (edozein $i, j \in \{1, \ldots, r\}$-tarako),

$$\omega(xy) \geq \omega(x) + \omega(y) \tag{A.3}$$

da. Ondorioz, $\mathfrak{U}^m$ ideala da.

(ii) Izan bedi $r \in R \setminus \{0\}$, orduan

$$rx \in \mathfrak{U}^m \implies \omega(x) = \omega(rx) > m \implies x \in \mathfrak{U}^m,$$

hau da, $\mathfrak{U}^m$ ideal isolatua da.

Bestalde, $\mathcal{U}_R(\mathfrak{L})/\mathfrak{U}^m$ finituki sortua da,

$$\mathcal{B}_m = \{\mathbf{x}^\alpha + \mathfrak{U}^m \mid \omega(\mathbf{x}^\alpha) \leq m\}$$

multzoak sortzen baitu. Hortaz, $\mathfrak{U}^m$ koheina finituko R-modulu aske isolatua da, eta, (A.3)ren arabera, $x \in \mathfrak{L}$ guztietarako $\ell_x(\mathfrak{U}^m) \subseteq \mathfrak{U}^m$ da. Beraz, edozein m-tarako ezker adierazpen erregularrak

$$\mathcal{L}_m \colon \mathfrak{L} \to \mathrm{End}_R\left({}^{\mathcal{U}_R(\mathfrak{L})}/_{\mathfrak{U}^m}\right), \; x \mapsto \ell_x$$

adierazpen finitua ematen du –izan bedi $f \in \mathrm{End}_R(\mathcal{U}_R(\mathfrak{L}))$ non $f(\mathfrak{X}) \subseteq \mathfrak{X}$ den $\mathfrak{X} \trianglelefteq \mathcal{U}_R(\mathfrak{L})$ ideal baterako, notazio abusu batekin, berriro f erabiliko da $x + \mathfrak{X} \mapsto f(x) + \mathfrak{X}$ moduan definituriko $f \in \mathrm{End}_R(\mathcal{U}_R(\mathfrak{L})/\mathfrak{X})$ endomorfismoa izendatzeko–.

Bestetik, $\mathfrak{L} \cap \mathfrak{U}^c(\mathfrak{L}) = \{0\}$ da. Izan ere, $x = \sum_{i=1}^r \alpha_i x_i$ bada,

$$\omega(x) = \omega\left(\sum_{i=1}^r \alpha_i x_i\right) \leq \max_{i=1,\ldots,r} \omega(x_i) = c$$

da.

Hori dela eta, $\mathcal{L}_c$ $\mathfrak{L}$-ren adierazpen leial finitua da, eta horren maila bornatzeko, nahikoa da $|\mathcal{B}_c|$ goitik bornatzea. Alde batetik, $\mathcal{B}_c$-n dauden monomio guztiek gehienez c pisua dute, eta, beraz, gehienez c polinomio maila. Bestetik, r aldagaitan gehienez c mailako monomio kopurua $r+1$ aldagaitan c mailako monomio kopurua da (aldagai laguntzaile bat gehituz homogeneizatu monomio guztiak zehatz-mehatz c maila izan dezaten).

Lema A.10. *r aldagaitan c mailako monomio kopurua $\binom{r+c-1}{c}$ da.*

Beraz, borne laño hau dugu (konparatu [31, Korolarioa 5.1]):

$$\deg \mathfrak{L} \leq \operatorname{rk}\left(\frac{\mathcal{U}_R(\mathfrak{L})}{\mathfrak{U}^c(\mathfrak{L})}\right) \leq \binom{r+c}{c}.$$

Atal hau bukatzeko $\deg \mathfrak{L}$-ren borne ez oso zorrotz bat emanen dugu, baina soilik $\operatorname{rk} \mathfrak{L}$-ren menpe. Izan ere, $c \in \{1, \ldots, r\}$ da, hau da, $c = \alpha r$ da $\alpha \in \{1/r, \ldots, r-1/r, 1\}$ izanik. Bestetik, Stirlingen hurbilketa formularen arabera,

$$\sqrt{2\pi r}\,(r/e)^r \leq r! \leq \sqrt{2\pi r}\,(r/e)^r\, e^{\frac{1}{12r}} \ \forall r \in \mathbb{N}.$$

da. Ondorioz,

$$\begin{aligned}\binom{r+\alpha r}{\alpha r} &\leq \frac{\sqrt{2\pi(r+\alpha r)}(r+\alpha r)^{r+\alpha r}}{\sqrt{2\pi\alpha r}(\alpha r)^{\alpha r}\sqrt{2\pi r}\ r^r} e^{\frac{1}{12(1+\alpha)r}} \\ &\leq \frac{e^{1/12(1+\alpha)r}}{\sqrt{2\pi r}} \frac{\sqrt{1+\alpha}}{\sqrt{\alpha}} \left(\frac{(1+\alpha)^{1+\alpha}}{\alpha^\alpha}\right)^r.\end{aligned}$$

Gainera, bi ezberdintzetan ezkerreko eta eskuineko terminoak asintotikoki baliokideak dira, hau da, haien zatidura 1era doa r infinitura joan ahala. Halaber, $\alpha \in [1/r, 1]$ denez,

$$\frac{\sqrt{1+\alpha}}{\sqrt{\alpha}} \leq \sqrt{r+1} \text{ eta } \frac{(1+\alpha)^{1+\alpha}}{\alpha^\alpha} \leq 4$$

dira. Hots,

$$\frac{e^{1/12(1+\alpha)r}}{\sqrt{2\pi}}\frac{1}{\sqrt{r}}\frac{\sqrt{1+\alpha}}{\sqrt{\alpha}}\left(\frac{(1+\alpha)^{1+\alpha}}{\alpha^{\alpha}}\right)^{r}\leq\sqrt{\frac{r+1}{r}}4^{r}.$$

Horrenbestez,

$$\binom{r+c}{c}\leq\sqrt{\frac{r+1}{r}}4^{r}\quad\forall c\in\{1,\ldots,r\}.\tag{A.4}$$

A.3.2 Lieren erretikulu banangarriak

Bigarren urratsean R-Lie aljebra adierazpen egoki bat emanen dugu R-Lie erretikulu banangarri deituko ditugunetarako.

Ideal nilpotenteen batura berriro ideal nilpotentea da (konparatu [41, Kapitulua I, Proposizioa 7.6]), beraz, finituki sortutako $\mathfrak{L}$ R-Lie aljebra guztiek badute ideal nilpotente oro barruan duen ideal nilpotente bat, $\mathfrak{L}$-ren *erradikal nilpotente* deituko duguna eta $R_n(\mathfrak{L})$ adieraziko dena. Bereziki, $Z(\mathfrak{L})\leq R_n(\mathfrak{L})$ da, eta $R_n(\mathfrak{L})$ ideal isolatua da, $R_n(\mathfrak{L})=R_n(\mathfrak{L}_K)\cap\mathfrak{L}$ da eta.

Horrela, $\mathfrak{L}$ heina finituko R-Lie erretikulua *banangarria* dela diogu, existitzen bada $\mathfrak{S}\leq\mathfrak{L}$ R-Lie azpialjebra bat non $\mathfrak{L}=R_n(\mathfrak{L})\oplus\mathfrak{S}$ den, hau da, $\mathfrak{L}$ Lie aljebra $\mathfrak{S}$-ren biderketa erdizuzena da $R_n(\mathfrak{L})$ idealarekin. Kategoria teoriaren ikuspegitik baliokidea da esatea

$$0\to R_n(\mathfrak{L})\to\mathfrak{L}\to\mathfrak{L}/R_n(\mathfrak{L})\to 0$$

segida zehatz laburra banatu egiten dela R-Lie aljebren kategorian.

Kasu banangarrirako, aurreko adierazpen erregularra eta deribazioetatik eratorritako adierazpenak konbina daitezke:

Definizioa A.11. Izan bedi A R-aljebra. Orduan, $D\in\mathrm{End}_R(A)$ R-modulu endomorfimoa *deribazioa* da *Leibnizen identitatea* betetzen badu, hau da,

$$D(ab)=aD(b)+D(a)b\ \forall a,b\in A.$$

Deribazio guztien multzoa $\mathrm{Der}_R(A)$ izendatzen da.

Esate baterako, Jacobiren identitatearen eraginez, edozein $x\in\mathfrak{L}$-tarako $\mathrm{ad}_x\in\mathrm{Der}_R(\mathfrak{L})$ da. Bestalde, $\mathfrak{L}$ R-Lie erretikuluaren D deribaziotik abiatuta $\mathcal{U}_R(\mathfrak{L})$-ren deribazio bat eraiki daiteke, D^* deituko duguna, Leibnizen identitatea betetzea inposatuz. Hots,

$$D^*(u_1 \dots u_t) = \sum_{i=1}^{t} u_1 \dots u_{i-1} D(u_i) u_{i+1} \dots u_t$$

arauaren hedapen lineala hartu eta $D^*(1) = 0$ definitu (identitatedun aljebra guztietarako bete behar baitu azken horrek). Hedadura hori propietate unibertsalaren kasu partikularra baino ez da (konparatu [41, Kapitulua V, Teorema 1.1(7)]).

Lema A.12. *Izan bitez $\mathfrak{L}$ R-Lie erretikulu nilpotentea eta $D \in \mathrm{Der}_R(\mathfrak{L})$. Orduan, $D^*(\mathfrak{U}^m(\mathfrak{L})) \subseteq \mathfrak{U}^m(\mathfrak{L})$ da $m \in \mathbb{N}$ guztietarako.*

Froga. Demagun ω pisu funtzioa $\mathfrak{L}$-ren $\{x_1, \dots, x_r\}$ R-oinarriarekiko definitu dela (konparatu Azpiatala A.3.1). Orduan, D R-Lie aljebra homomorfismoa denez, $D(\mathfrak{L}_i) \subseteq \mathfrak{L}_i$ da $i \in \{1, \dots, c\}$ guztietarako, eta, beraz, $\omega(D(x_i)) \geq \omega(x_i)$ da $i \in \{1, \dots, r\}$ guztietarako, hau da, oinarriko elementu guztietarako. Horrenbestez, $x_{i_1} \dots x_{i_t} \in \mathfrak{U}^m$ bada, (A.3) dela eta,

$$\omega\left(D^*(x_{i_1} \dots x_{i_t})\right) = \min_{j=1,\dots,t} \{\omega(x_{i_1} \dots D(x_j) \dots x_{i_t})\} \geq \omega(x_{i_1} \dots x_{i_t}) > m. \quad \square$$

Teorema A.13 (Zassenhaussen hedapena, cf. [11, Kapitulua I, § 7.3, Teorema 1] eta [41, Kapitulua VI, Teorema 2.1])**.** *Izan bedi $\mathfrak{L}$ R-Lie erretikulu banangarria eta izan bedi c zenbaki osoa $R_n(\mathfrak{L})$-ren nilpotentzia klasea. Orduan, existitzen da $\mathfrak{L}$-ren*

$$\Phi \colon \mathfrak{L} \to \mathrm{End}_R\left(\frac{\mathcal{U}_R(R_n(\mathfrak{L}))}{\mathfrak{U}^c(R_n(\mathfrak{L}))}\right)$$

adierazpena zeina injektiboa den $R_n(\mathfrak{L})$-n. Bereziki, $\deg \Phi$ *soilik* $\mathrm{rk}\, R_n(\mathfrak{L})$*-ren menpekoa da.*

Froga. Izendatu $\mathfrak{N} := R_n(\mathfrak{L})$. Orduan, $\mathfrak{L} = \mathfrak{N} \oplus \mathfrak{S}$ da, $\mathfrak{S} \leq \mathfrak{L}$ R-Lie azpialjebra baterako, eta Lema A.12 dela eta, $\mathrm{ad}_x^*\left(\mathfrak{U}^c(\mathfrak{N})\right) \subseteq \mathfrak{U}^c(\mathfrak{N})$ da $x \in \mathfrak{L}$ guztietarako. Hortaz, definitu

$$\Phi \colon \mathfrak{L} = \mathfrak{N} \oplus \mathfrak{S} \to \mathrm{End}_R\left({}^{\mathcal{U}_R(\mathfrak{N})}/{}_{\mathfrak{U}^c(\mathfrak{N})}\right), \; n + s \mapsto \ell_n + \mathrm{ad}_s^* .$$

funtzioa. Ikus dezagun Φ R-Lie aljebra homomorfismoa dela. Horretarako nahikoa da

$$\Phi\left([s, n]\right) = [\Phi(s), \Phi(n)] = [\mathrm{ad}_s^*, \ell_n]$$

dela ikustea $n \in \mathfrak{N}$ eta $s \in \mathfrak{S}$ guztietarako. Alde batetik, ohartu $n \in \mathfrak{N}$ eta $D \in \mathrm{Der}_R\left(\mathcal{U}_R(\mathfrak{N})\right)$ guztietarako

$$[D, \ell_n](u) = D \circ \ell_n(u) - \ell_n \circ D(u) = D(n)u = \ell_{D(n)}(u) \ \forall u \in \mathfrak{N}$$

dela. Bestalde, $\mathfrak{N}$ ideala denez, $[s, n] \in \mathfrak{N}$ da, eta, horrenbestez,

$$\Phi\left([s, n]\right) = \ell_{[s,n]} = \ell_{\mathrm{ad}_s^*(n)} = [\mathrm{ad}_s^*, \ell_n] = [\Phi(s), \Phi(n)] \,.$$

Beraz, Φ R-Lie aljebra adierazpena da, eta haren nukleoak ebakidura tribiala du erradikal nilpotentearekin, $\Phi|_{\mathfrak{N}} = \mathcal{L}_c$ eta $\mathcal{L}_c$ adierazpen leiala baitira.

Bukatzeko, identitatea (A.4)ren arabera,

$$\deg \Phi \leq \sqrt{\frac{\mathrm{rk}\, R_n(\mathfrak{L}) + 1}{\mathrm{rk}\, R_n(\mathfrak{L})}} \cdot 4^{\mathrm{rk}\, R_n(\mathfrak{L})} \tag{A.5}$$

da. □

A.3.3 Murgilketa teorema

Nilpotentzia bezala, R-Lie aljebretan ebazgarritasuna defini daiteke: $\mathfrak{L}$ R-Lie aljebraren *serie deribatua* errekurtsiboki

$$\mathfrak{L}^{(1)} := \mathfrak{L},\ \mathfrak{L}^{(i)} := \left[\mathfrak{L}^{(i-1)}, \mathfrak{L}^{(i-1)}\right] \ \forall i \geq 2$$

moduan definitzen da, eta $\mathfrak{L}$ *ebazgarria* da existitzen bada $\ell \in \mathbb{N}$ zenbaki osoa non $\mathfrak{L}^{(\ell)} = \{0\}$ den. Hori betetzen duten zenbaki osoetan txikiena da, hain zuzen ere, $\mathfrak{L}$-ren *deribazio luzera*. Are gehiago, ideal ebazgarrien batura berriro ere ebazgarria da (konparatu [41, Kapitulua I, Proposizioa 7.4]), eta, horrenbestez, finituki sortutako $\mathfrak{L}$ Lie aljebraren *erradikal ebazgarria* defini daiteke, hau da, beste ideal ebazgarri oro barruan duen $R_s(\mathfrak{L})$ ideal ebazgarria. Noski, aljebra nilpotente guztiak ebazgarriak dira eta, ondorioz, $R_n(\mathfrak{L}) \leq R_s(\mathfrak{L})$ da.

Leviren Teoremaren arabera (ikusi [41, Kapitulua III, Atala 9]), R zero karakteristikako gorputza denean, $\mathfrak{L}$ R-Lie aljebra oro $R_s(\mathfrak{L}) \oplus \mathfrak{S}$ moduan deskonposatzen da, $\mathfrak{S} \leq \mathfrak{L}$ R-Lie azpialjebra semisinple bat delarik, *Leviren osagarri* deitua. Eskuarki deskonposizio horrek garrantzia handia du gorputzen gainean Adoren Teorema frogatzerakoan. Izan ere, lehenbiziko $R_s(\mathfrak{L})$-ren adierazpen leial finitu bat eraiki ohi da, eta ondotik adierazpen hori $\mathfrak{L}$-ra hedatu, Zassenhaussen hedapenaren bidez (konparatu Teorema A.13).

Aitzitik, R-Lie aljebra orokorretan Leviren Teorema ez da beti betetzen. Adibidez, hartu $\mathfrak{sl}_2(2\mathbb{Z}) \oplus \mathfrak{t}_2(2\mathbb{Z})$ $\mathbb{Z}$-Lie aljebra, hau da, $2\mathbb{Z}$ eraztunaren gainean 2×2 tamainako eta 0 aztarnako matrizeen eta 2×2 tamainako matrize goi triangeluarren aljebren arteko batura zuzena, ez da bilatutako moduan deskonposatzen (ikusi [19, 838. orriko adibidea]).

Hirugarren eta azken urrats honetan teorema orokorra frogatuko dugu. Horretarako, hasierako $\mathfrak{L}$ R-Lie erretikulua R-Lie erretikulu banangarri batean murgilduko dugu, eta ondoren aurreko azpiatala aplikatu. Azken batean, problema emaitza honetara murrizten da:

Proposizioa A.14 (cf. [19, Teorema 2ren frogako 2. urratsa])**.** *Izan bedi $\mathfrak{L}$ R-Lie erretikulua. Demagun $\mathfrak{L}_K$ K-Lie aljebra banangarri batean murgiltzen dela. Orduan, $\mathfrak{L}$ R-Lie erretikulu banangarri batean, deitu $\bar{\mathfrak{L}}$, murgiltzen da, eta*

(i) $R_n(\mathfrak{L}) \leq R_n(\bar{\mathfrak{L}})$ *eta*

(ii) $\operatorname{rk} R_n(\mathfrak{L}) = \operatorname{rk} R_n(\bar{\mathfrak{L}})$

dira.

Froga. Demagun $\mathfrak{L}_K$ K-Lie aljebra $\mathfrak{L}'_K = \mathfrak{N}'_K \oplus \mathfrak{S}'_K$ erretikulu banangarrian murgiltzen dela, non $\mathfrak{N}'_K = R_n(\mathfrak{L}'_K)$ den. Ohartu bide batez $R_n(\mathfrak{L}) \subseteq R_n(\mathfrak{L}_K) \subseteq R_n(\mathfrak{L}'_K)$ dela.

Izan bedi $\mathfrak{S}$ $\mathfrak{L}$-ren proiekzioa $\mathfrak{S}'_K$-ra, hau da,

$$\mathfrak{S} = \{s \in \mathfrak{S}'_K \mid \exists\, x \in \mathfrak{L} \text{ non } x = n + s \text{ den } n \in \mathfrak{N}'_K \text{ baterako}\}\,.$$

Erraz frogatzen da $\mathfrak{S}$ R-Lie aljebra dela. Izan ere, izan bitez $x_1 = n_1 + s_1$ eta $x_2 = n_2 + s_2 \in \mathfrak{L}$, non $n_i \in \mathfrak{N}'_K$ eta $s_i \in \mathfrak{S}'_K$ ($i \in \{1, 2\}$). Orduan,

$$[x_1, x_2] = [n_1, x_2] + [s_1, n_2] + [s_1, s_2] \tag{A.6}$$

da, eta $[x_1, x_2] \in \mathfrak{L}$, $[n_1, x_2] + [s_1, n_2] \in \mathfrak{N}'_K$ eta $[s_1, s_2] \in \mathfrak{S}'_K$ dira. Hori dela eta, $\mathfrak{S}$ finituki sortua denez, existitzen da $\lambda \in R \setminus \{0\}$ non $\lambda\mathfrak{S} \subseteq \mathfrak{L}$ den.

Izan bitez $s = \operatorname{rk} R_n(\mathfrak{L})$ eta c $\mathfrak{N}'_K$-ren nilpotencia klasea. Definitu $\mathfrak{N}_i := \gamma_i(\mathfrak{N}'_K) \cap \mathfrak{L} \trianglelefteq \mathfrak{L}$, $i \in \{1, \ldots, c\}$, ideal isolatuak. Horrela, $\mathfrak{N}_i \trianglelefteq \mathfrak{N}_1 = R_n(\mathfrak{L})$ da; eta, beraz, $R_n(\mathfrak{L})$-ren $\{x_1, \ldots, x_s\}$ R-oinarri bat aukera dezakegu non lehenengo s_1 elementuak $\mathfrak{N}_c$-ren oinarria diren, lehenengo s_2 elementuak ($s_2 \geq s_1$) $\mathfrak{N}_{c-1}$-en oinarria diren, etab.

Definitu orain

$$\mathfrak{N} := \bigoplus_{i=1}^{c} \frac{1}{\lambda^i}\mathfrak{N}_i \subseteq R_n(\mathfrak{L}_K)$$

R-modulua. Orduan, $\mathfrak{N}$ s heinako R-modulu askea da,

$$\left\{ \frac{1}{\lambda^i} x_j \mid 1 \leq i \leq c,\ s_{c-i} \leq j \leq s_{c-i+1} \right\}$$

oinarria baitu (hemen $s_0 = 1$ eta $s_c = s$ dira). Gainera,

$$\left[\frac{1}{\lambda^i}\mathfrak{N}_i, \frac{1}{\lambda^j}\mathfrak{N}_j \right] = \frac{1}{\lambda^{i+j}} \left[\mathfrak{N}_i, \mathfrak{N}_j\right] \leq \frac{1}{\lambda^{i+j}}\mathfrak{N}_{i+j}$$

denez, $\mathfrak{N}$ R-Lie erretikulu nilpotentea da.

Hartu, hondarrik, $\bar{\mathfrak{L}} := \mathfrak{N} \rtimes \mathfrak{S}$. Ikus dezagun $\bar{\mathfrak{L}}$ R-Lie erretikulua dela, eta $\mathfrak{L}$ barruan duela. Alde batetik, $\mathfrak{S} = \mathfrak{L} + {}^1\!/\!_\lambda \mathfrak{N}_1$ da eta, ondorioz, $\mathfrak{L} \subseteq \bar{\mathfrak{L}}$ da. Gainera,

$$\begin{aligned} \left[\frac{1}{\lambda^i}\mathfrak{N}_i, \mathfrak{S} \right] &\leq \left[\frac{1}{\lambda^i}\mathfrak{N}_i, \mathfrak{L} + \frac{1}{\lambda}\mathfrak{N}_1 \right] \leq \frac{1}{\lambda^i}\left[\mathfrak{N}_i, \mathfrak{L}\right] + \frac{1}{\lambda^{i+1}}\left[\mathfrak{N}_i, \mathfrak{N}_1\right] \\ &\leq \frac{1}{\lambda^i}\mathfrak{N}_i + \frac{1}{\lambda^{i+1}}\mathfrak{N}_{i+1} \leq \mathfrak{N} \end{aligned}$$

da $i \in \{1, \ldots, c\}$ guztietarako. Beraz, $\bar{\mathfrak{L}}$ R-Lie erretikulua da. Azkenik, $\bar{\mathfrak{L}}$-ren erradikal nilpotentea $\mathfrak{N}$ da, eta, horrenbestez, $R_n(\mathfrak{L}) \leq R_n(\bar{\mathfrak{L}})$, $\operatorname{rk} R_n(\bar{\mathfrak{L}}) = \operatorname{rk} R_n(\mathfrak{L}) = s$ eta eraikuntzaz $\bar{\mathfrak{L}}$ banangarria da. □

Orobat, Leviren Teoremaren arabera, R zero karakteristikako gorputza denean $\mathfrak{L}$ R-Lie aljebra banangarria da $R_n(\mathfrak{L}) = R_s(\mathfrak{L})$ bada. Ideal nagusietako domeinuen gainean, ordea, egoera ez dago argi. Halere, $\mathfrak{L}$ R-Lie erretikulu banangarri batean murgildu daiteke, alegia:

Korolarioa A.15. *Izan bedi $\mathfrak{L}$ R-Lie erretikulua eta demagun $R_n(\mathfrak{L}) = R_s(\mathfrak{L})$ dela. Orduan, existitzen da $\mathfrak{L}$ hedatzen duen R-Lie erretikulu banangarri bat, deitu $\bar{\mathfrak{L}}$, non $R_n(\mathfrak{L}) \leq R_n(\bar{\mathfrak{L}})$ eta $\operatorname{rk} R_n(\mathfrak{L}) = \operatorname{rk} R_n(\bar{\mathfrak{L}})$ diren.*

Froga. Leviren Teoremaren arabera, $R_n(\mathfrak{L}_K) = R_s(\mathfrak{L}_K)$ denez, $\mathfrak{L}_K$ banangarria da. □

Azkenik, bila genbiltzan murgilketa teorema dugu, Neretinek frogatu zuena [60] (hurrengo frogak Lieren aljebrako emaitza sakonak darabiltza, eta hemen gainetik aipatu baino ez ditugu eginen).

Teorema A.16 (Murgilketa Teorema, cf. [60]). *Izan bedi $\mathfrak{L}$ R-Lie erretikulua. Existitzen da $\bar{\mathfrak{L}}$ R-Lie erretikulua $\mathfrak{L}$ hedatuz non $R_n(\mathfrak{L}) \leq R_n(\bar{\mathfrak{L}})$ eta* $\operatorname{rk} R_n(\bar{\mathfrak{L}}) = \operatorname{rk} R_s(\mathfrak{L})$ *diren.*

Froga. Proposizioa A.14 kontuan hartuta, nahikoa da $\mathfrak{L}_K$ hedatzen duen $\mathfrak{L}'_K$ K-Lie algebra banangarria eraikitzea $\dim_K R_n(\mathfrak{L}'_K) = \operatorname{rk} R_s(\mathfrak{L})$ izanik.

Demagun $R_n(\mathfrak{L}) = \langle x_1, \ldots, x_s \rangle_R$ dela eta izan bedi $\mathfrak{S}_K = \langle z_1, \ldots, z_l \rangle_K$ $\mathfrak{L}_K$-ren Leviren faktore bat. Orduan, badaude $y_1, \ldots, y_k \in R_s(\mathfrak{L}_K)$ elementu batzuk, $k = \operatorname{rk} R_s(\mathfrak{L}) - s$, non

$$\mathfrak{L}_K = Kx_1 \oplus \cdots \oplus Kx_s \oplus Kz_1 \oplus \cdots \oplus Kz_l \oplus Ky_1 \oplus \cdots \oplus Ky_k$$

den. Gainera, [41, Kapitulua II, Teorema 7.13]ren arabera,

$$[R_s(\mathfrak{L}_K), \mathfrak{L}_K] \leq R_n(\mathfrak{L}_K) \tag{A.7}$$

da. Horrenbestez,

$$\mathfrak{I}_i := \langle x_1, \ldots, x_s, z_1, \ldots, z_h \rangle \oplus Ky_1 \oplus \cdots \oplus Ky_i$$

ideala da $i \in \{0, \ldots, k\}$ guztietarako. Bereziki, $\mathfrak{L}_K = \mathfrak{I}_k$ $\mathfrak{S}_K$-aljebra da. Halaber, $\mathfrak{S}_K$ semisinplea eta $\operatorname{char} K = 0$ direnez, Weylen Erreduzibilitate Osoaren Teorema dela eta (ikusi [41, Kapitulua III, Teorema 7.8]), $\mathfrak{L}_K = \mathfrak{I}_k$ $\mathfrak{S}_K$-modulu semisinplea da, eta existitzen da $K\tilde{y}_k$ $\mathfrak{S}_K$-modulu nagusia non

$$\mathfrak{L}_K = \mathfrak{I}_{k-1} \oplus K\tilde{y}_k$$

den. Horrela, $K\tilde{y}_k$ $\mathfrak{S}_K$-modulua da, baina aldi berean (A.7) dela eta, $[K\tilde{y}_k, \mathfrak{S}_K] \leq R_n(\mathfrak{L}_K)$ da, eta, horrenbestez, $[K\tilde{y}_k, \mathfrak{S}_K] = 0$ da.

Bestetik, $\operatorname{ad}_{\tilde{y}_1} \in \operatorname{Der}_K(\mathfrak{L}_K)$ eta K zero karakteristikakoa direnez, Jordan-Chevalleyren deskonposizioaren arabera (ikusi [41, Kapitulua III, Teorema 11.17] eta [63, Proposizioa 3]), existitzen dira $d_{n,1}$ deribazio nilpotentea (i.e. $d_{n,1}^C = 0$, $C \in \mathbb{N}$ baterako) eta $d_{s,1}$ deribazio semisinplea (i.e. $d_{s,1}$ eragilea diagonalizagarria da K^{alg} itxitura aljebraikoaren gainean) non

$$\operatorname{ad}_{\tilde{y}_1} = d_{n,1} + d_{s,1}$$

den. Are gehiago, deribazio horiek bakarrak dira eta elkarrekin trukatzen dira.

Izan bedi K-Lie aljebra hau:

$$\mathfrak{L}_1 = \langle x_1, \ldots, x_s, x'_1 \rangle_K \oplus \langle z_1, \ldots, z_l, z'_1 \rangle_K,$$

x_1' eta z_1' sinbolo formal hutsak izanik, Lieren kortxete honekin:

$$[u,v]_{\mathfrak{L}_1} = [u,v]_{\mathfrak{L}_K} \qquad [u,x_1']_{\mathfrak{L}_1} = d_{n,1}(u)$$

$$[u,z_1']_{\mathfrak{L}_1} = d_{s,1}(u) \qquad \text{eta } [x_1',z_1']_{\mathfrak{L}_1} = 0,$$

$u,v \in R_n(\mathfrak{L}_k) \rtimes \mathfrak{S}_K$ guztietarako.

Ohartu $\tilde{y}_1 = x_1' + z_1'$ identifikazioak $\mathfrak{I}_1 \hookrightarrow \mathfrak{L}_1$ Lie aljebra monomorfismoa definitzen duela.

Gainera, $\mathrm{ad}_{x_1'}$ deribazio nilpotentea denez,

$$\mathfrak{N}_1 = \langle x_1, \ldots, x_s, x_1' \rangle = R_n(\mathfrak{L}_K) \oplus K x_1'$$

Lie aljebra nilpotentea da. Gainera, $\tilde{y}_1$ eta $\mathfrak{S}_K$ elkarrekin trukatzen direnez, $[z_1', \mathfrak{S}_K] = 0$ da eta, beraz,

$$\mathfrak{S}_1 = \mathfrak{S}_K \oplus K z_1'$$

Lie aljebra erreduktiboa da, Lie aljebra semisinple eta abeldar baten batuketa zuzena baita. Bestalde, $\mathrm{ad}_{z_1'}$ eragile semisinplea denez eta $\mathfrak{S}_K$-ren ekintza $\mathfrak{L}_1$-ren gainean guztiz erreduzigarria denez, $\mathfrak{S}_1$-en ekintza $\mathfrak{L}_1$-n ere guztiz erreduzigarria da. Bereziki, $\mathfrak{I}_{k-2} \trianglelefteq \mathfrak{I}_{k-1} \trianglelefteq \mathfrak{L}_1$ denez, badago $K\tilde{y}_{k-1}$ $\mathfrak{S}_1$-modulu nagusia non

$$\mathfrak{I}_{k-1} = \mathfrak{I}_{k-2} \oplus K\tilde{y}_{k-1}$$

den.

Errepikatu aurreko argudioa $\mathrm{ad}_{\tilde{y}_{k-1}} \in \mathrm{Der}_K(\mathfrak{L}_1)$ deribazioarekin, hots, existitzen da

$$\mathfrak{L}_2 := \langle x_1, \ldots, x_s, x_1', x_2' \rangle_K \oplus \langle z_1, \ldots, z_l, z_1', z_2' \rangle_K$$

K-Lie aljebra bat. Gainera, $\mathfrak{N}_2 = \mathfrak{N}_1 \oplus K x_2'$ nilpotentea da, $\mathfrak{S}_2 = \mathfrak{S}_1 \oplus K z_2'$ erreduktiboa da eta ekintza guztiz erreduzigarria du $\mathfrak{L}_2$-n.

Prozedura hau k aldiz errepikatuz, azkenean $\mathfrak{L}_k = \mathfrak{N}_k \oplus \mathfrak{S}_k$ aljebra banangarria erdiesten da, non $\mathfrak{N}_k = \langle x_1, \ldots, x_s, x_1', \ldots, x_k' \rangle_K$ erradikal nilpotentea den eta $\mathfrak{S}_k = \langle z_1, \ldots, z_l, z_1', \ldots, z_k' \rangle_K$ K-Lie aljebra erreduktiboa. Azkenik, $\mathfrak{L}_k$ Lie aljebrak $\mathfrak{L}_K$ hedatzen du $\tilde{y}_j = x_j' + z_j'$ identifikazioekin, eta $\dim_K R_n(\mathfrak{L}_K') = r + k = \mathrm{rk}\, R_s(\mathfrak{L})$ da. $\square$

Horrenbestez, osagai guztiak bildu daitezke emaitza nagusia frogatzeko:

Teorema A.4ren froga. Izan bedi $\mathfrak{L}$ r heinako R-Lie erretikulua. Orduan, existitzen da $\mathfrak{L}$ hedatzen duen $\bar{\mathfrak{L}}$ R-Lie erretilu banangarria non $R_n(\mathfrak{L}) \leq R_n(\bar{\mathfrak{L}})$ eta $\operatorname{rk} R_n(\bar{\mathfrak{L}}) = \operatorname{rk} R_s(\mathfrak{L})$ diren. Teorema A.13 dela eta, badago $\bar{\mathfrak{L}}$-ren Φ R-Lie aljebra adierazpena zeina injektiboa den $R_n(\bar{\mathfrak{L}})$-n eta Φ-ren maila $f(\operatorname{rk} R_n(\bar{\mathfrak{L}}))$-k bornatzen du $f\colon \mathbb{N}_0 \to \mathbb{N}$, $r \mapsto \sqrt{\frac{r+1}{r}}4^r$ funtzio ez-beherakorrerako (ikusi (A.5)).

Hortaz, $\tilde{\Phi} := \Phi|_{\mathfrak{L}} \oplus \operatorname{Ad}$ adierazpena $\mathfrak{L}$-ren R-Lie adierazpen leiala da,

$$\ker \tilde{\Phi} = \ker \Phi|_{\mathfrak{L}} \cap \ker \operatorname{Ad} \subseteq (\mathfrak{L} \setminus R_n(\mathfrak{L})) \cap Z(\mathfrak{L}) = \{0\}$$

baita. Hori dela eta, $\operatorname{rk} R_n\left(\bar{\mathfrak{L}}\right) = \operatorname{rk} R_s(\mathfrak{L})$ denez, orduan

$$\deg \mathfrak{L} \leq \deg \tilde{\Phi} \leq f(\operatorname{rk} R_s(\mathfrak{L})) + r \leq f(r) + r = \sqrt{\frac{r+1}{r}}4^r + r$$

da. □

A.4 Oharrak

Apendizeko emaitza originalak [77]n biltzen dira.

Bukatzeko ohar pare bat egin behar ditugu. Lehenik eta behin, Harish-Chandrak [34]n, Adoren Teoremaren beste froga bat eman zuen gorputzen gainean, eta emaitza apur bat hobetu zuen, erradikal nilpotenteko elementuak endomorfimo nilpotentetara –$f \in \operatorname{End}_R(W)$ *endomorfimo nilpotentea* da existitzen bada $C \in \mathbb{N}_0$ non $f^C = 0$ den– bidaltzen dituen adierazpen leial bat existitzen dela frogatu baitzuen. Horrelako adierazpenei *nil-adierazpen* deritze. Ohartu, ordea, aitzineko frogan eraikitako $\tilde{\Phi} = \Phi \oplus \operatorname{Ad}$ adierazpena nil-adierazpena dela, adierazpen adjuntua eta Teorema A.13ko Φ adierazpena nil-adierazpenak baitira. Azken batean, edozein $x \in R_n(\mathfrak{L})$-rako

$$x^{c+1} \in \mathfrak{U}^c(R_n(\mathfrak{L}))$$

da, eta, ondorioz,

$$\Phi(x)^{c+1} \equiv \ell_x^{c+1} = \ell_{x^{c+1}} \equiv 0 \mod \mathfrak{U}^c(R_n(\mathfrak{L}))$$

da.

Bigarrenik, argudio konbinatorio garatuagoak erabilita

$$\operatorname{rk}\left(\frac{\mathcal{U}_R(\mathfrak{L})}{\mathfrak{U}^c(\mathfrak{L})}\right) < \alpha\frac{2^r}{\sqrt{r}}$$

dela frogatu daiteke, non

$$\alpha = \sqrt{\frac{2}{\pi}} \prod_{j=1}^{\infty} \frac{2^j}{2^j - 1} \in (2.762, 2.763)$$

balio erreala den (ikusi [13, Lemma 5 (3)]). Hots, ideal nagusietako domeinuetarako gorputzetarako ezagutzen zen bornerik onena, (A.1) borena, berreskuratu daiteke. Hala ere, gure helburuetarako, nagusiki 2. kapituluan erabiltzea, Teorema A.4ko bornea nahikoa da.

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Colophon

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