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**SUPERCORE FOR NORMAL FORM
GAMES**

by

Elena Iñarra, Concepción Larrea and Ana Saracho

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The Supercore for Normal Form Games[‡]

ELENA INARRA[‡], CONCEPCION LARREA, and ANA SARACHO

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Abstract

We study the supercore of a system derived from a normal form game. For the case of a finite game with pure strategies, we define a sequence of games and show that the supercore of that system coincides with the set of Nash equilibrium strategy profiles of the last game in the sequence. This result is illustrated with the characterization of the supercore for the n -person prisoners' dilemma. With regard to the mixed extension of a normal form game, we show that the set of Nash equilibrium profiles coincides with the supercore for games with a finite number of Nash equilibria. For games with an infinite number of Nash equilibria this need not be no longer the case. Yet, it is not difficult to find a binary relation which guarantees the coincidence of these two sets.

Keywords: Individual contingent threat situation, Nash equilibrium, sub-solution, von Neumann and Morgenstern stable set.

JEL classification: C70.

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[†]Universidad del Pais Vasco-Euskal-Herriko Unibertsitatea. E-mails: jepingae@bs.ehu.es (E. Inarra), elplajac@bs.ehu.es (C. Larrea) and jepsadea@bs.ehu.es (A. Saracho).

[‡]Correspondence address: Elena Inarra, Universidad del Pais Vasco, Avda. Lehendakari Agirre 83, 48015 Bilbao, Spain.

1 Introduction

Stable sets were first defined by von Neumann and Morgenstern (1947) as a solution to n -person cooperative games and have received a substantial deal of attention in the literature of games (see, for instance, Lucas (1992) and another references therein). A recent approach to the stable set theory and its connections with other solution concepts in game theory has been developed in the book “The Theory of Social Situations” (TOSS) by Greenberg (1990). In its Chapter 7 it is argued that when modeling social environments, normal form games do not capture the notion of negotiation among players, while an advantage of the approach proposed in TOSS is that the consequences of different types of negotiations among players may be analyzed. One of the negotiations considered by Greenberg is the so called individual contingent threat (ICT) situation, where each single player can object to the prevailing outcome and can threaten the others by stating that she will use a different strategy.¹ The ICT situation can be used to generate a system in which the strategy profiles of a normal form game are the elements of the set and the binary relation defined on that set accounts only for the profitable single deviations.

With respect to the existence of von Neumann and Morgenstern (vN&M) stable sets for systems associated to an ICT situation of a finite normal form game, Greenberg shows that it is guaranteed in the following two cases: (i) when there are at most two players, and (ii) when there are n players, each one with a set of at most two strategies.² Unfortunately, however, these existence theorems cannot be generalized for any number of players or strategies. Even in the case of games with Nash equilibrium (NE) strategy profiles (Nash (1951)), the existence of a vN&M stable set is not assured.³ Along this line of research, Arce (1994) studies the vN&M stable sets for a 3-person prisoners’ dilemma and Nakanishi (2001, 2002) shows the existence of vN&M stable sets for the n -person prisoners’ dilemma with continuous strategies and also for some other games.

¹Negotiations where players can jointly object to the prevailing outcome and can threaten the others by stating that they will use another strategies are considered by Greenberg (1989, 1990) and Kahn and Mookherjee (1992).

²Greenberg (1990), pp. 100-101, Theorems 7.4.5 and 7.4.6.

³Greenberg (1990), p. 102, Example 7.4.8.

Subsolutions were defined by Roth (1976) as a generalization of the vN&M stable sets. Interestingly enough, this notion has not been extensively considered in the game theory literature. However, the supercore is a distinct subsolution and does have a better performance than the vN&M stable sets for the particular setting considered in this paper. Thus, our objective is to analyze the supercore as a solution for systems associated to an ICT situation of a finite normal form game in pure strategies and of its mixed extension.

For the pure strategies case, the supercore of the system associated to an ICT situation contains at least the NE strategy profiles. In particular, given a normal form game we derive a sequence of games and we find that the set of NE strategy profiles of the last game in the sequence exactly coincides with the supercore of the system associated to the original game. As a result, this procedure allows the identification of those games in which the supercore selects exactly the NE strategy profiles. With regard to the content of the supercore, this solution may be interpreted as the outcome of a dynamic model of sequential selection of strategy profiles. From this perspective, the supercore is formed by the union of NE and the “NE protected strategy profiles” of each game in the sequence.

We illustrate the previous results with a numerical example and we also characterize the supercore of the system associated to the n -person prisoners’ dilemma. In the last case, the supercore is the unique vN&M stable set and it is formed by the strategy profile where all players choose to defect and by the strategy profiles in which the number of players who choose to cooperate is even.

For the case of a system associated to the mixed extension of a finite normal form game the results are noteworthy. One of the first criticisms of the von Neumann and Morgenstern (vN&M) stable sets was made by Harsanyi (1974) who argued that this notion is unsatisfactory because it neglects the destabilizing effect of the indirect dominance relation between alternatives of the stable set. As we shall show this shortcoming is not shared by the supercore when applying to *generic* games. More precisely, we find that the supercore *exactly coincides* with the set of NE strategy profiles for normal form games that have a

finite number of NE profiles. However, a simple example shows that this result no longer holds for the case of games with an infinite number of NE strategy profiles.

To obtain the equivalence between the two sets we proceed in a parallel fashion to Kalai and Schmeidler (1977). These authors find, under the same binary relation we have considered in this paper, that the admissible set may be “too large” since it may coincide with the entire space of mixed strategies. They also show the coincidence of the admissible set and the set of NE under a suitable dominance relation. As indicated above in our case, the equivalence between the supercore and the set of NE strategy profiles is obtained for the mixed extensions of *almost all* normal form games. Yet, it is not difficult to find a weaker binary relation which guarantees the exact coincidence of these two sets for the mixed extension of *every* normal form game.

The rest of the paper is organized as follows. Section 2 contains the preliminaries. In Section 3 we introduce the ICT situation associated to a normal form game, we define the sequence of normal form games which allows the determination of the supercore, and we present the characterization of the supercore for the n -person prisoners’ dilemma. Section 4 studies the supercore associated to the mixed extension of a game under two different binary relations. An appendix with several proofs concludes the paper.

2 Preliminaries

These preliminaries introduce the solution concepts for an abstract system that will be used in the paper, and also recall the definitions of a finite normal form game, its mixed extension and the Nash equilibrium solution.

An *abstract system* is a pair (X, R) , where X is a set of elements and R is an irreflexive binary relation defined on X . The relation R reads “dominates.” Hence, if for two elements x, x' in X we have xRx' , then we say that x dominates x' .

For any $x \in X$, let $\mathcal{D}(x)$ denote the dominion of x , *i.e.*, $\mathcal{D}(x) = \{x' \in X : xRx'\}$. Given any subset A of X , we define the following sets: $\mathcal{D}(A) = \bigcup_{x \in A} \mathcal{D}(x)$, $\mathcal{U}(A) = X - \mathcal{D}(A)$,⁴ and $\mathcal{P}(A) = \mathcal{U}(A) - A$.

A *subsolution* of (X, R) is a subset A of X such that $A \subset \mathcal{U}(A)$,⁵ and $A = \mathcal{U}^2(A)$, where $\mathcal{U}^2(A) = \mathcal{U}(\mathcal{U}(A))$. The condition $A \subset \mathcal{U}(A)$ is known as the internal stability condition. With regard to the condition $A = \mathcal{U}^2(A)$, Roth (1976, p. 44) provides the following interpretation:

.. every point in $\mathcal{U}(A) - A$ is dominated by some other point in the same set and the entire set, thus “overrules” itself leaving only the set as “sound.”

In words, if A is a subsolution then $\mathcal{P}(A) \subset \mathcal{D}(\mathcal{P}(A))$.

The intersection of all subsolutions of (X, R) is also a subsolution which is known as the *supercore*.

A subset $A \subset X$ is a *vN&M stable set* of (X, R) if $A = \mathcal{U}(A)$. Thus, a vN&M stable set is characterized by the internal stability condition $A \subset \mathcal{U}(A)$, and by $\mathcal{U}(A) \subset A$, known as the external stability condition. Clearly, a vN&M stable set is a subsolution that satisfies $\mathcal{P}(A) = \emptyset$.

A subset $A \subset X$ is the *core* of (X, R) if $A = \mathcal{U}(X)$.

A finite *normal form game* Γ^N is a triple $\langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ where $N = \{1, \dots, n\}$ is the finite set of players, S_i is the finite set of strategies for player i and $u_i : S = \times_{i \in N} S_i \rightarrow \mathbb{R}$ is player i 's payoff function.

⁴The symbol $-$ stands for the difference binary relation.

⁵The symbol \subset means weakly contained while \subsetneq means strictly contained.

A strategy of player i , \widehat{s}_i is a best response to s_{-i} if for all $s_i \in S_i$, $u_i(\widehat{s}_i, s_{-i}) \geq u_i(s_i, s_{-i})$ where $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$.

Let $s = (s_1, \dots, s_n)$ denote a strategy profile. Then, $s^* = (s_1^*, \dots, s_n^*)$ is a *Nash equilibrium* in Γ^N if s_i^* is a best response to s_{-i}^* for all $i \in N$.

A *mixed extension of the game* Γ^N is a triple $\langle N, \{\Delta S_i\}_{i \in N}, \{U_i\}_{i \in N} \rangle$ where ΔS_i is the simplex of the mixed strategies for player i , and $U_i : \Delta(S) = \times_{i \in N} \Delta S_i \rightarrow \mathbb{R}$, assigns to $\sigma \in \Delta(S)$ the expected value under u_i of the lottery over S that is induced by σ so that $U_i(\sigma) = \sum_{s \in S} (\prod_{j \in N} \sigma_j(s_j)) u_i(s)$.

Let σ denote a mixed strategy profile. Then, $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ is a *Nash equilibrium* in the mixed extension of the game Γ^N if σ_i^* is a best response to $\sigma_{-i}^* = (\sigma_1^*, \dots, \sigma_{i-1}^*, \sigma_{i+1}^*, \dots, \sigma_n^*)$ for all $i \in N$.

3 The Supercore of a Finite Normal Form Game in Pure Strategies

This section has 3 subsections. In the first one, we define the system associated to an ICT situation of a normal form game. In the second we define a sequence of normal form games that allows the determination of the supercore of the system associated to an ICT situation of a normal form game in pure strategies. The third subsection concludes with a numerical example that illustrates these results and also includes the characterization of the supercore for the n -person prisoners' dilemma.

3.1 A System Associated to an ICT Situation of a Finite Normal Form Game

The application of the approach developed in TOSS to the normal form of a game allows the consideration of different types of negotiation among players, and their possible consequences. In particular, the negotiation in which each player may deviate from a proposed strategy profile unilaterally is the one that we study in this paper. This notion of negotiation is formalized by ICT situation. Let us start with a description of the negotiation procedure:

A strategy profile, say s , is proposed to players. If all individuals openly consent to follow s , then s will be adopted. If player i objects to s , then she has

to openly declare that if the remaining players stick to the specified profile s , then she will choose s'_i instead of s_i (contingent threat of player i). Thus, each single player can object to the prevailing profile and can threaten the others by saying that she will choose another strategy. We say that player i induces s' from s when she modifies profile s into profile s' . Any player other than player i can then counter the new upcoming strategy profile induced by player i . The set of profiles that player i can induce from s is denoted as:

$$\gamma_i(s) = \{s' \in S : s'_j = s_j \text{ for all } j \neq i, j \in N\}.$$

Thus, γ_i determines an inducement correspondence for player i from S into itself. Once we have γ_i we can define the ICT situation associated with Γ^N as:

$$\Gamma_\gamma^N \equiv (N, S, \{u_i\}_{i \in N}, \{\gamma_i\}_{i \in N}).$$

We are now ready to define the system associated to an ICT situation of a game in normal form.

Definition 1 *An individual dominance system associated to Γ_γ^N is a pair (S, \angle) , where \angle is the binary relation defined on S as follows:*

$$s' \angle s \text{ if there exists } i \in N \text{ such that } s' \in \gamma_i(s) \text{ and } u_i(s') > u_i(s).$$

This means that s' dominates s if s' is derived from s via a deviation of a player i who is better off under s' than under s .

(Hereafter, the individual dominance system will be simply called as the system whenever no confusion is possible)

3.2 A Procedure to Compute the Supercore of (S, \angle)

Let us consider a game Γ^N with at least one NE strategy profile. This assumption is not restrictive since the supercore of (S, \angle) for a game Γ^N with no NE strategy profile is the empty set (Roth, (1976)).

Let S^* be the set of NE strategy profiles of the game Γ^N and let $s^* \in S^*$. Starting from s^* , consider the set of strategy profiles obtained by a deviation of a player who obtains a lower payoff than the payoff provided by s^* . This set is the dominion of s^* , that is, $\mathcal{D}(s^*) = \{s \in S : s \in \gamma_i(s^*) \text{ and } u_i(s^*) > u_i(s) \text{ for}$

some $i \in N$. Hence, it is clear that moving from s into s^* is always profitable for player i . Thus, the dominion of S^* is $\mathcal{D}(S^*) = \bigcup_{s^* \in S^*} \mathcal{D}(s^*)$.

Let $v_i(\Gamma^N)$ be the lowest payoff for player i in the game Γ^N . That is, $v_i(\Gamma^N) = \min\{u_i(s) : s \in S\}$. Denote by $v(\Gamma^N) = (v_1(\Gamma^N), \dots, v_n(\Gamma^N))$ the vector of lowest payoffs.

In what follows we offer a procedure to determine the supercore of (S, \angle) . The basic intuition for this procedure is the following. Starting from the game Γ^N , we define a new game Γ_1^N with the same set of players and strategies for every player and with the players' payoffs modified in the following way: the payoff for each player at every profile in $\mathcal{D}(S^*)$ is equal to his lowest payoff in the game Γ^N , while the payoffs corresponding to the remaining strategy profiles are maintained. The idea behind this modification is to take any power away from the strategy profiles dominated by the NE strategy profiles. By assigning them the lowest payoffs of the game, these strategy profiles cannot longer dominate any profile.

With this modification in hand, we may verify whether or not game Γ_1^N has any additional NE strategy profiles than those that game Γ^N has. If Γ_1^N has new NE strategy profiles then a game Γ_2^N can be defined, and the procedure may continue iteratively in this manner.

The procedure described generates a sequence of games $\langle \Gamma_t^N \rangle_{t=0}^\infty$ and a sequence of systems $\langle (S, \angle_t) \rangle_{t=0}^\infty$ defined inductively as follows:

(i) $\Gamma_0^N = \Gamma^N$ and $(S, \angle_0) = (S, \angle)$.

(ii) For $t \geq 1$, $\Gamma_t^N = \langle N, \{S_i\}_{i \in N}, \{u_i^t\}_{i \in N} \rangle$, where:

$$u_i^t(s) = \begin{cases} v_i(\Gamma^N) & \text{if } s \in \mathcal{D}(S_{t-1}^*) \text{ in } (S, \angle_{t-1}) \\ u_i^{t-1}(s) & \text{otherwise,} \end{cases}$$

where S_{t-1}^* denotes the set of NE strategy profiles of Γ_{t-1}^N , and (S, \angle_t) is the associated system to Γ_t^N in which \angle_t is the binary relation on S given by:

$$s' \angle_t s \text{ if there is a player } i \in N \text{ such that } s' \in \gamma_i(s) \text{ and } u_i^t(s') > u_i^t(s).$$

Formally, this procedure can be summarized as follows:

Step 0: Let $\Gamma_0^N = \Gamma^N$. Compute S_0^* and determine $\mathcal{D}(S_0^*)$ in (S, \angle_0) . Game Γ_1^N is generated according to the player's payoff function $\{u_i^1\}_{i \in N}$, and the system (S, \angle_1) is generated by the relation \angle_1 .

Step t: Let be the game Γ_t^N . Compute S_t^* .

If $S_t^* = S_{t-1}^*$, then the procedure concludes.

If $S_{t-1}^* \subsetneq S_t^*$, determine $\mathcal{D}(S_t^*)$ in (S, \angle_t) . Game Γ_{t+1}^N is generated according to the player's payoff function $\{u_i^{t+1}\}_{i \in N}$, and the system (S, \angle_{t+1}) is generated by the relation \angle_{t+1} .

Given that S is finite, there exists a $k \in \mathbb{N}$ such that $S_t^* \neq S_{t+1}^*$ for all $t = 0, \dots, k-2$ and $S_k^* = S_{k-1}^*$.

Now, we can establish the following theorem:

Theorem 1 *Let S_k^* be the set of NE strategy profiles of the game Γ_k^N . Then S_k^* is the supcore of (S, \angle) .*

Proof. We will prove that the following two conditions hold:

- (i) S_k^* is a subsolution of (S, \angle) .
- (ii) Any other subsolution \bar{S} of (S, \angle) contains S_k^* .
 - (i) We show that S_k^* satisfies in (S, \angle) the conditions $S_k^* \subset \mathcal{U}(S_k^*)$ and $S_k^* = \mathcal{U}^2(S_k^*)$.

Given the way the sequence of games $\langle \Gamma_0^N, \dots, \Gamma_k^N \rangle$ is constructed, we can write that for each $s \in S$ and for all $i \in N$

$$u_i^k(s) = \begin{cases} v_i(\Gamma^N) & \text{if } s \in \mathcal{D}(S_k^*) \text{ in } (S, \angle) \\ u_i(s) & \text{otherwise.} \end{cases} \quad (1)$$

Clearly, the NE strategy profiles of Γ_k^N in the system (S, \angle) cannot dominate each other and can only be dominated by the strategy profiles belonging to $\mathcal{D}(S_k^*)$. Hence, $S_k^* \subset \mathcal{U}(S_k^*)$ and $S_k^* \subset \mathcal{U}(\mathcal{U}(S_k^*))$. Now, let us assume that there is a strategy profile $s \in \mathcal{U}(\mathcal{U}(S_k^*))$ such that $s \notin S_k^*$. Then s will be dominated in (S, \angle_k) by some $s' \in S$. Therefore $s' \angle_k s$, and from (1) it follows that $s' \notin \mathcal{D}(S_k^*)$ in (S, \angle) . Since $s, s' \notin \mathcal{D}(S_k^*)$ the players' payoffs corresponding to the profiles s and s' are the same in the games Γ_k^N and Γ^N , it follows that $s' \angle s$.

Therefore, $s \in \mathcal{D}(\mathcal{U}(S_k^*))$, which contradicts that $s \in \mathcal{U}(\mathcal{U}(S_k^*))$. Consequently, $S_k^* = \mathcal{U}(\mathcal{U}(S_k^*))$.

(ii) We argue by contradiction. Suppose that for some subsolution \bar{S} of (S, \angle) , $S_k^* \not\subset \bar{S}$. Now, let us consider $S_0^* \subset \dots \subset S_k^*$ and define $l = \min\{t : S_t^* \not\subset \bar{S}, t = 0, \dots, k\}$. Since S_0^* is the core of (S, \angle) , it is included in any subsolution. Therefore, $l \neq 0$.

Let $s \in S_l^*$ such that $s \notin \bar{S}$. Then, either $s \in \mathcal{D}(\bar{S})$ or $s \in \mathcal{P}(\bar{S})$ in (S, \angle) . Given that s is a Nash equilibrium in Γ_l^N , it can only be dominated by some strategy profiles in $\mathcal{D}(S_{l-1}^*)$ and, by the definition of l , we have that \bar{S} is a subsolution such that $S_{l-1}^* \subset \bar{S}$ so that $s \notin \mathcal{D}(\bar{S})$. Hence, $s \in \mathcal{P}(\bar{S})$. Now, given that $\mathcal{P}(\bar{S}) \subset \mathcal{D}(\mathcal{P}(\bar{S}))$, there exists $s' \in \mathcal{P}(\bar{S})$ such that $s' \angle s$. However, given that $s' \in \mathcal{D}(S_{l-1}^*)$ and $S_{l-1}^* \subset \bar{S}$ then $s' \notin \mathcal{P}(\bar{S})$. Thus, we have reached a contradiction. ■

Corollary 1 *The core of (S, \angle) coincides with the supercore of (S, \angle) if and only if $S_0^* = S_1^*$.*

Proof. Given that S_0^* is the core of (S, \angle) the result follows directly from Theorem 1. ■

One question that readily arises is the type of profiles included in the supercore. The content of this set for (S, \angle) may be interpreted as the result of a dynamic model of sequential selection of strategy profiles.⁶ Starting from Γ_0^N and S_0^* , assume that the strategy profiles in $\mathcal{D}(S_0^*)$ “lose power,” so that the payoffs of the players in these profiles are replaced by their corresponding lowest payoffs in the game. By taking into account these “lowered payoffs”, we determine Γ_1^N , which is a game where the profiles in $\mathcal{D}(S_0^*)$ cannot dominate any profile. Thus, starting from Γ_1^N , we determine the set of NE profiles S_1^* . Clearly the strategy profiles belonging to $S_1^* - S_0^*$ are dominated only by some profiles in $\mathcal{D}(S_0^*)$. We call them “NE protected profiles”.

In general, given the game Γ_t^N , $t = 1, \dots, k$, S_t^* is formed by the set S_{t-1}^* and by those NE profiles protected by S_{t-1}^* . Therefore, we may establish that the supercore is formed by S_0^* and the NE protected profiles of each game in the rest of the sequence.

⁶See the dynamic model presented by Roth (1978).

3.3 Two Examples

We first present a simple example illustrating some previous results and then characterize the supercore for the n -person prisoners' dilemma.

Example 1. Consider the following game Γ^N :

	b_1	b_2	b_3	b_4
a_1	6,6	5,5	1,3	2,2
a_2	3,4	4,4	7,2	1,3
a_3	6,2	2,3	8,8	6,2
a_4	2,3	2,5	9,4	2,5

Step 0: Let $\Gamma_0^N = \Gamma^N$. The vector of lowest payoffs is $(v_1(\Gamma^N), v_2(\Gamma^N)) = (1, 2)$. The set of NE strategy profiles of Γ_0^N is $S_0^* = \{(a_1, b_1)\}$ and the dominion of S_0^* is $\mathcal{D}(S_0^*) = \{(a_1, b_2), (a_1, b_3), (a_1, b_4), (a_2, b_1), (a_4, b_1)\}$. Replacing the payoffs of the profiles in $\mathcal{D}(S_0^*)$ by $(1, 2)$ game Γ_1^N is obtained.

Step 1: Let the game Γ_1^N be:

	b_1	b_2	b_3	b_4
a_1	6,6	1,2	1,2	1,2
a_2	1,2	4,4	7,2	1,3
a_3	6,2	2,3	8,8	6,2
a_4	1,2	2,5	9,4	2,5

Here, we have $S_1^* = \{(a_1, b_1), (a_2, b_2)\}$, and $\mathcal{D}(\{(a_1, b_1), (a_2, b_2)\}) = \mathcal{D}(\{(a_1, b_1)\}) \cup \{(a_2, b_3), (a_2, b_4), (a_3, b_2), (a_4, b_2)\}$. Replacing the payoffs of the profiles in $\mathcal{D}(S_1^*)$ by $(1, 2)$ game Γ_2^N is obtained.

Step 2: Let the game Γ_2^N be:

	b_1	b_2	b_3	b_4
a_1	6,6	1,2	1,2	1,2
a_2	1,2	4,4	1,2	1,2
a_3	6,2	1,2	8,8	6,2
a_4	1,2	1,2	9,4	2,5

The set of NE strategy profiles of Γ_2^N is $S_2^* = \{(a_1, b_1), (a_2, b_2)\}$. Since $S_2^* = S_1^*$ the procedure ends.

This procedure generates the sequence of games $\langle \Gamma_0^N, \Gamma_1^N, \Gamma_2^N \rangle$. The set of NE strategy profiles of the game Γ_2^N is the supercore for (S, \angle) . The two vN&M stable sets of the system (S, \angle) are $\{(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4)\}$ and $\{(a_1, b_1), (a_2, b_2), (a_4, b_3), (a_3, b_4)\}$.

Example 2 *The Supercore for the n-person Prisoners' Dilemma:*⁷

The n -person prisoners' dilemma represents situations where the cooperative outcome, all players selecting cooperation, cannot be attained as a NE. Let us formally define this game.⁸ Let N be the set of players. Assume that every player has two actions C (cooperation) and D (defection). The payoff of player i is given by:

$$f_i(a|r), a = C, D, \text{ and } r = 0, \dots, n-1,$$

where a is player i 's action and r is the number of other players who select action C .

The following three assumptions on the payoff function define an n -person prisoners' dilemma:

- A.1 Every player is better off by choosing D than by choosing C . That is, for all $i \in N$: $f_i(C|r) < f_i(D|r)$ for all $r = 0, \dots, n-1$.
- A.2 If all players choose D , then the payoff for each and every player is worse than the payoff they would obtain if they all chose C . That is, for all $i \in N$: $f_i(C|n-1) > f_i(D|0)$.
- A.3 The payoff of player i , given her action, increases as the number of other players that select C increases; that is, $f_i(C|r)$ and $f_i(D|r)$ are increasing functions of r .

Under these assumptions it is straightforward to see that the unique NE strategy profile is that in which all players select D .

Let (S_{pd}, \angle) denote the system associated to the n -person prisoners' dilemma.

⁷Arce (1994) studies the vN&M stable set for a 3-person prisoners' dilemma. Using sets of continuous strategies for all players Nakanishi (2001) shows that a vN&M stable set always exists for an n -person prisoner's dilemma.

⁸Here, we follow Nishihara's (1997) formulation of the n -person prisoners' dilemma. See also Okada (1993).

Theorem 2 *The supercore of the n -prisoners' dilemma is the unique vN&M stable set of (S_{pd}, \angle) , and it is formed by (D, \dots, D) and by those strategy profiles such that the number of players who choose C is even.*

Proof. Using the procedure described above we have the sequence of games $\langle \Gamma_0^N, \dots, \Gamma_k^N \rangle$ that is derived in the following way. *Step 0:* Let $\Gamma_0^N = \Gamma^N$. The set of NE strategy profiles is $S_0^* = \{(D, \dots, D)\}$ and by A.1 the dominion of S_0^* is $\mathcal{D}(S_0^*) = \{s \in S_{pd} : s_i = C \text{ and } s_j = D, \text{ for all } j \neq i.\}$. *Step t ($t \geq 1$):* Take the game Γ_t^N . The set of NE strategy profiles is $S_t^* = S_{t-1}^* \cup \tilde{S}_t$ where \tilde{S}_t is the set of profiles such that exactly $2t$ players choose C . The dominion of S_t^* is $\mathcal{D}(S_t^*) = \mathcal{D}(S_{t-1}^*) \cup \mathcal{D}(\tilde{S}_t)$ where $\mathcal{D}(\tilde{S}_t)$ is the set of strategy profiles such that exactly $(2t + 1)$ players choose C . *Step k :* Since $S_{k-1}^* = S_k^*$, it must happen that $k = \frac{n}{2} + 1$ if n is even, and $k = \text{integer part of } \frac{n}{2} \text{ plus } 1$ if n is odd. It is clear that $S_{pd} = S_k^* \cup \mathcal{D}(S_k^*)$. Hence, we may conclude that the supercore of (S_{pd}, \angle) is a vN&M stable set and it is obviously unique. ■

Lastly, we conclude this section with two comments:

1. A drawback of the supercore when applied to this setting is that, in general, it may not include any efficient strategy profile, that is those profiles in which there is no other strategy profile where all players are strictly better off. Example 1 illustrates the case where the supercore does not include any of the two efficient profiles, (a_3, b_3) and (a_4, b_3) . In the n -person prisoners' dilemma game, however, the inclusion in the supercore of some efficient strategy profiles is guaranteed. Notice that: (i) if the number of players in the game is even then the strategy profile (C, \dots, C) is in the supercore, and (ii) if the number of players is odd then all those profiles with exactly one player choosing D are in the supercore. It is easy to see that A.1, A.2 and A.3 guarantee that (C, \dots, C) and the strategy profiles in which exactly one player chooses D are efficient.

2. As it is well known, a strategy is rationalizable if it survives the iterated removal of strategies that are never best response (see Bernheim (1984) and Pearce (1984)). It turns out that the selected strategy profiles by the supercore are not always rationalizable. For example, the supercore for the 2-person prisoners' dilemma is the set $\{(D, D), (C, C)\}$, while D is the only rationalizable strategy.

4 The Supercore of the Mixed Extension of a Finite Normal Form Game

In this section we study the supercore of a system associated to the mixed extension of a normal form game. We find that if the number of NE profiles is finite then the supercore of the system coincides with the set of NE of the mixed extension of the game. A simple example shows that this is no longer the case if the number of NE strategy profiles is infinite.

An *ICT situation of the mixed extension of a game* Γ^N is a 4-tuple $\langle N, \Delta(S), (U_i)_{i \in N}, (\gamma_i)_{i \in N} \rangle$ where γ_i is the correspondence from $\Delta(S)$ into itself defined by

$$\gamma_i(\sigma) = \{\sigma' \in \Delta(S) : \sigma'_j = \sigma_j \text{ for all } j \neq i, j \in N\}.$$

Thus, $\gamma_i(\sigma)$ is the set of profiles which may be induced by player i from σ .

Definition 2 An *individual dominance system associated to an ICT situation of the mixed extension of a game* Γ^N is a pair $(\Delta(S), \angle)$ where \angle is the binary relation defined on $\Delta(S)$ such that:

$$\sigma' \angle \sigma \text{ if there exists } i \in N \text{ such that } \sigma' \in \gamma_i(\sigma) \text{ and } U_i(\sigma') > U_i(\sigma).$$

Let Σ^* be the set of NE strategy profiles of the mixed extension of the game Γ^N and let $\sigma^* \in \Sigma^*$. The dominion of σ^* is $\mathcal{D}(\sigma^*) = \{\sigma \in \Delta(S) : \sigma \in \gamma_i(\sigma^*) \text{ and } U_i(\sigma^*) > U_i(\sigma) \text{ for some } i \in N\}$. Then, the dominion of Σ^* will be $\mathcal{D}(\Sigma^*) = \bigcup_{\sigma^* \in \Sigma^*} \mathcal{D}(\sigma^*)$.

Theorem 3 *If Σ^* is finite then Σ^* is the supercore of $(\Delta(S), \angle)$.*

Proof. We first prove that Σ^* is a subsolution of $(\Delta(S), \angle)$, that is that $\Sigma^* \subset \mathcal{U}(\Sigma^*)$ and $\Sigma^* = \mathcal{U}^2(\Sigma^*)$.

Given that $\Sigma^* \subset \mathcal{U}(\Sigma^*)$, if $\Sigma^* = \mathcal{U}(\Sigma^*)$ then $\Sigma^* = \mathcal{U}^2(\Sigma^*)$ and Σ^* is a subsolution. If $\Sigma^* \neq \mathcal{U}(\Sigma^*)$ we have to show that $\mathcal{P}(\Sigma^*) \subset \mathcal{D}(\mathcal{P}(\Sigma^*))$, which is equivalent to the condition $\Sigma^* = \mathcal{U}^2(\Sigma^*)$ given that $\Sigma^* \subset \mathcal{U}^2(\Sigma^*)$.

Let $\sigma \in \mathcal{P}(\Sigma^*)$. We will show that $\sigma \in \mathcal{D}(\mathcal{P}(\Sigma^*))$.

Since $\sigma \notin \Sigma^*$ then σ_i will not be the best response to σ_{-i} for some player i . Therefore, there exists a profile $\sigma' \in \gamma_i(\sigma)$ such that $U_i(\sigma') > U_i(\sigma)$. Now,

set $\sigma_\lambda = \lambda\sigma + (1 - \lambda)\sigma'$ for all $\lambda \in [0, 1]$. By the linearity of U_i we have $U_i(\sigma_\lambda) > U_i(\sigma)$, and since $\sigma_\lambda \in \gamma_i(\sigma)$, it follows that $\sigma_\lambda \angle \sigma$ for all $\lambda \in [0, 1]$. Thus, σ_λ dominates σ , and $\sigma_\lambda \notin \Sigma^*$ given that $\sigma \in \mathcal{P}(\Sigma^*)$.

It remains to prove that $\sigma_\lambda \in \mathcal{P}(\Sigma^*)$ for some λ : If $\sigma_\lambda \in \mathcal{D}(\Sigma^*)$ for all λ , then there exists $\sigma_\lambda^* \in \Sigma^*$ such that $\sigma_\lambda^* \in \gamma_j(\sigma_\lambda)$ for some player j , and $U_j(\sigma_\lambda^*) > U_j(\sigma_\lambda)$. If $j = i$, we have $U_i(\sigma_\lambda^*) > U_i(\sigma_\lambda) > U_i(\sigma)$. Hence, σ_λ^* will dominate σ , which implies that $\sigma \in \mathcal{D}(\Sigma^*)$. Otherwise, the subset $\{\sigma_\lambda^* : \lambda \in [0, 1]\}$ of Σ^* will be infinite, which contradicts the fact that Σ^* is finite. Therefore, $\sigma_\lambda \in \mathcal{P}(\Sigma^*)$ for some λ , and since $\sigma_\lambda \angle \sigma$ it follows that $\sigma \in \mathcal{D}(\mathcal{P}(\Sigma^*))$.

Lastly, since the supercore is the intersection of all subsolutions and any subsolution contains Σ^* , Theorem 3 follows. ■

The result above does not longer hold when the mixed extension of the game Γ^N has an infinite number of NE strategy profiles. The following example illustrates that non-Nash equilibrium strategy profiles may belong to the supercore of $(\Delta(S), \angle)$.

Example 3 Consider the mixed extension of the following game:

	b_1	b_2
a_1	1,0	1,1
a_2	-1,1	1,0

Let p be the probability that player 1 chooses a_1 and let q be the probability that player 2 chooses b_1 . It is easy to check that the set of Nash equilibria is $\Sigma^* = \{(p, 1 - p, 0, 1) : \frac{1}{2} \leq p \leq 1\}$. The dominion of the set of Nash equilibria is $\mathcal{D}(\Sigma^*) = \{(p, 1 - p, q, 1 - q) : \frac{1}{2} < p \leq 1, 0 < q \leq 1\}$ and the set of profiles undominated by the set of Nash equilibria excluding them is $\mathcal{P}(\Sigma^*) = \{(p, 1 - p, q, 1 - q) : 0 \leq p \leq \frac{1}{2}, 0 < q \leq 1\} \cup \{(p, 1 - p, 0, 1) : 0 \leq p < \frac{1}{2}\}$. It is straightforward to show that the supercore of $(\Delta(S), \angle)$ is $\Sigma^* \cup \{(\frac{1}{2}, \frac{1}{2}, q, 1 - q) : 0 < q \leq 1\} \cup \{(p, 1 - p, 0, 1) : 0 \leq p < \frac{1}{2}\}$.

The example above suggests that the equivalence between the supercore and the set of NE strategy profiles for the mixed extension of a game might perhaps require the use of a weaker dominance relation than the one previously considered in this paper.

In this regard, it is interesting to consider the work by Kalai and Schmeidler (1977). These authors study the admissible set in various bargaining situations. In particular, they study the admissible set for the mixed extension of a game under different binary relations. Using the same binary relation we have considered in this paper, they find that the admissible set may be “too large” (for instance, in the 2-person matching pennies game the admissible set coincides with the entire space of mixed strategies) and show that the coincidence of the admissible set and the set of NE profiles holds under a somewhat different binary relation. This relation embeds the notion of *possible reply*, where each player rationalizes her reply taking into account the possible rationalization of the remaining players.

In our case, as we have shown in Theorem 3, the equivalence of the supercore and the set of NE profiles is obtained for mixed extensions of *almost all* normal form games (in particular it holds for generic games). To obtain the exact coincidence between these two sets we introduce a new dominance relation.

Definition 3 Let $(\Delta(S), \ll)$ be the weakly individual dominance system associated to the mixed extension of the game Γ^N , where \ll is the binary relation defined on $\Delta(S)$ as follows: $\sigma' \ll \sigma$ if there exists a player $i \in N$ such that $\sigma' \in \gamma_i(\sigma)$ and either $U_i(\sigma') > U_i(\sigma)$ or $U_i(\sigma') = U_i(\sigma)$ whenever $\sigma' \in \Sigma^*$ and $\sigma \notin \Sigma^*$.

This dominance relation between profiles may intuitively be justified as follows: As with the earlier relation, any *feasible* and *strictly* profitable deviation between strategy profiles will always occur. In addition, $U_i(\sigma') = U_i(\sigma)$ for $\sigma' \in \Sigma^*$, $\sigma \notin \Sigma^*$ means that starting from a non-NE profile, a player when facing a feasible NE profile that gives her the *same payoff*, prefers to deviate to that NE profile where she is assuring herself her current payoff. This may happen because she knows that at the non-NE profile, there is at least one player who will surely deviate to another profile, and this move might give her a lower payoff than the current one. Consequently, this lack of farsightedness about future payoffs implies that an equality in the payoffs between non-NE strategy profiles is not a sufficient reason for a player to deviate.

We establish two lemmas before showing the equivalence between the set of NE and the supercore.

Lemma 1 Σ^* is a compact subset of $\Delta(S)$.

Proof. See the appendix. ■

Lemma 2 $\mathcal{D}(\Sigma^*) \cup \Sigma^*$ in $(\Delta(S), <<)$ is a closed subset of $\Delta(S)$.

Proof. See the appendix. ■

Finally, we will show the equivalence between the set of NE strategy profiles and the supercore of $(\Delta(S), <<)$.

Theorem 4 Σ^* is the supercore of $(\Delta(S), <<)$.

Proof. See the appendix. ■

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Appendix

Proofs omitted from the text are provided below.

Proof of Lemma 1

We first prove that Σ^* is closed.

Let us consider a sequence $\{\sigma_n^*\}_{n \in \mathbb{N}} \subset \Sigma^*$ such that $\lim_{n \rightarrow \infty} \sigma_n^* = \sigma^*$. We will see that $\sigma^* \in \Sigma^*$. Since $\sigma_n^* \in \Sigma^*$ then $(\sigma_n^*)_i$ is a best response to $(\sigma_n^*)_{-i}$ for each $i \in N$. That is,

$$U_i((\sigma_n^*)_i, (\sigma_n^*)_{-i}) \geq U_i(\sigma_i, (\sigma_n^*)_{-i}) \text{ for all } \sigma_i \in \Delta(S_i).$$

Taking the limit on each side of the last expression we have:

$$\lim_{n \rightarrow \infty} U_i((\sigma_n^*)_i, (\sigma_n^*)_{-i}) \geq \lim_{n \rightarrow \infty} U_i(\sigma_i, (\sigma_n^*)_{-i}) \text{ for all } \sigma_i \in \Delta(S_i).$$

Since $\lim_{n \rightarrow \infty} \sigma_n^* = \sigma^*$, and U_i is a continuous function it follows that:

$$U_i((\sigma^*)_i, (\sigma^*)_{-i}) \geq U_i(\sigma_i, (\sigma^*)_{-i}) \text{ for all } \sigma_i \in \Delta(S_i).$$

Therefore, σ_i^* is player's i best response to σ_{-i}^* for every $i \in N$. In other words, σ^* is a NE strategy profile. Lastly, given that Σ^* is a closed subset of the compact set $\Delta(S)$ we conclude that Σ^* is compact. ■

Proof of Lemma 2

By Lemma 1, Σ^* is closed. Hence, it is sufficient to prove that the closure of $\mathcal{D}(\Sigma^*)$ is contained in $\mathcal{D}(\Sigma^*) \cup \Sigma^*$.

Let us consider a sequence $\{\sigma_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(\Sigma^*)$ such that $\lim_{n \rightarrow \infty} \sigma_n = \sigma$. We will see that $\sigma \in \mathcal{D}(\Sigma^*) \cup \Sigma^*$.

Since $\sigma_n \in \mathcal{D}(\Sigma^*)$, there is a NE strategy profile σ_n^* such that for some player $i \in N$, $\sigma_n^* \in \gamma_i(\sigma_n)$ and $U_i(\sigma_n^*) \geq U_i(\sigma_n)$. Taking into account that the set Σ^* is compact (Lemma 1) and that $\{\sigma_n^*\}_{n \in \mathbb{N}} \subset \Sigma^*$, we can assume without loss of generality the existence of a profile $\sigma^* \in \Sigma^*$ such that $\lim_{n \rightarrow \infty} \sigma_n^* = \sigma^*$. (If this is not the case then we replace that sequence by the appropriate subsequence).

Now, set $\mathbb{N}(i) = \{n \in \mathbb{N} : \sigma_n^* \in \gamma_i(\sigma_n)\}$ for each $i \in N$. It is clear that for some $j \in N$ the set $\mathbb{N}(j)$ is countable. Hence, we can choose the subsequences $\{\sigma'_n\}_{n \in \mathbb{N}}$ of $\{\sigma_n\}_{n \in \mathbb{N}}$ and $\{\sigma^{*'}_n\}_{n \in \mathbb{N}}$ of $\{\sigma_n^*\}_{n \in \mathbb{N}}$ such that $(\sigma^*)'_n \in \gamma_j(\sigma'_n)$ and

$U_j((\sigma^*)_n) \geq U_j(\sigma'_n)$ for all $n \in \mathbb{N}$. Therefore, taking the limit on each side in the last expression we have:

$$\lim_{n \rightarrow \infty} U_j((\sigma^*)_n) \geq \lim_{n \rightarrow \infty} U_j(\sigma'_n).$$

Since $\lim_{n \rightarrow \infty} ((\sigma^*)_n) = \sigma^*$, $\lim_{n \rightarrow \infty} (\sigma'_n) = \sigma$, and U_j is a continuous function, we have $U_j(\sigma^*) \geq U_j(\sigma)$. Given that $\sigma^* \in \gamma_j(\sigma)$ it follows that if $\sigma \notin \Sigma^*$ then $\sigma^* \ll \sigma$, so either $\sigma \in \mathcal{D}(\Sigma^*)$ or $\sigma \in \Sigma^*$. Thus, Lemma 2 follows. ■

Proof of Theorem 4

Given that any subsolution of $(\Delta(S), \ll)$ contains Σ^* , it is sufficient to prove that Σ^* is a subsolution. That is, $\Sigma^* \subset \mathcal{U}(\Sigma^*)$ and $\Sigma^* = \mathcal{U}^2(\Sigma^*)$.

Clearly, $\Sigma^* \subset \mathcal{U}(\Sigma^*)$. If $\Sigma^* = \mathcal{U}(\Sigma^*)$ then Σ^* is a vN&M stable set, and thus Σ^* is a subsolution. So, let us assume that $\mathcal{P}(\Sigma^*) \neq \emptyset$. We must show that $\Sigma^* = \mathcal{U}^2(\Sigma^*)$ or equivalently that $\mathcal{P}(\Sigma^*) \subset \mathcal{D}(\mathcal{P}(\Sigma^*))$.

Let $\sigma \in \mathcal{P}(\Sigma^*)$. Since $\sigma \notin \Sigma^*$, σ_i is not the best response to σ_{-i} for some player i . Hence, there is a $\sigma' \in \gamma_i(\sigma)$ such that $U_i(\sigma') > U_i(\sigma)$.

Now, if $\sigma' \in \mathcal{P}(\Sigma^*)$ then we obtain the desired result. If this is not the case then, set $\sigma_\lambda = \lambda\sigma + (1 - \lambda)\sigma'$ for all $\lambda \in [0, 1)$. By the linearity of U_i we have that $U_i(\sigma_\lambda) > U_i(\sigma)$, and since $\sigma_\lambda \in \gamma_i(\sigma)$, it follows that $\sigma_\lambda \ll \sigma$.

By Lemma 2 we know that $\mathcal{D}(\Sigma^*) \cup \Sigma^*$ is a closed subset of $\Delta(S)$. Therefore, $\mathcal{P}(\Sigma^*)$ is an open subset of $\Delta(S)$. This implies that there exists an $\varepsilon > 0$ such that the open ball $B(\sigma, \varepsilon) \subset \mathcal{P}(\Sigma^*)$. By choosing a $\lambda \in (0, 1)$ such that $\sigma_\lambda \in B(\sigma, \varepsilon)$ we have that $\sigma_\lambda \in \mathcal{P}(\Sigma^*)$. Since $\sigma_\lambda \ll \sigma$, we conclude that $\sigma \in \mathcal{D}(\mathcal{P}(\Sigma^*))$ and Theorem 4 yields. ■