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Some characterizations of the tensor product of complete lattices with applications to quantales

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Abstract This paper examines tensor products of complete lattices in which one factor is completely distributive. At least five characterizations of complete distributivity involving tensor products of complete lattices are given, among them this one: M is a completely distributive lattice if and only if for every complete lattice L the tensor product $M \otimes L$ is order isomorphic to the partially ordered set of all join- and meet-reversing maps from the complete lattice of all upclosed subsets of L to the lattice M. Some of these characterizations are then applied to give explicit descriptions of the multiplication of the tensor product of two quantales one of which is completely distributive.

Keywords Complete distributivity · tensor product · quantale

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1 Introduction

A few definitions have been proposed in the literature to describe the tensor product of the category Sup of complete lattices and join-preserving maps (cf. [1,7,12]).

In this paper we study the case in which one factor of the tensor product is completely distributive and we provide new descriptions of the tensor product in such a case. This approach leads to characterizations of complete distributivity based on the tensor product in Sup — e.g. complete distributivity is characterized via a description of joins in the tensor product $M \otimes L = [M, L^{op}]^{op}$ where $[M, L^{op}]^{op}$ is the complete lattice (ordered pointwisely) of all join-reversing maps from M to L (Theorem 1).

Further, let $\mathsf{Up}(L)$ be the complete lattice of all upclosed subsets of L provided with the containment relation $\subseteq^{op}$. By the universal property of the tensor product, the obvious bimorphism from $M \times L$ to $\mathsf{Up}(L) \otimes M$ – sending a pair (t, a) of $M \times L$ to the elementary tensor $(\uparrow a) \otimes t$ of $\mathsf{Up}(L) \otimes M$ – has a unique extension to a join-preserving map Φ from $M \otimes L$ to $\mathsf{Up}(L) \otimes M$. The map Φ appears to be injective for each complete lattice L if and only if M is completely distributive. This is crucial for the statement that M is a completely distributive lattice if and only if for every complete lattice L the map Φ determines an order isomorphism between $M \otimes L$ and the partially ordered set of all join- and meet-reversing maps from $\mathsf{Up}(L)$ to M (Theorem 2). The latter result and some of its consequences are then used to present a measure-theoretical description of tensors of $[0, +\infty]^{op} \otimes L$. Furthermore, if M is completely distributive, then there is an order isomorphism between $M^{op} \otimes L$ and $[M, L]$ (Fact 1).

We also consider the tensor product of quantales M and L with one factor being completely distributive and give explicit description of the multiplication in $M \otimes L$ (Theorem 3). The special case when M is the non-negative extended real half-line and L is the real unit interval, provides a new insight into some aspect of left-continuous quasi-inverse functions of [11].

2 Preliminaries

Let M and L be complete lattices with at least two different elements. Then $[M, L]$ always denotes the complete lattice of all join-preserving maps $M \xrightarrow{f} L$ which is ordered pointwisely — i.e.

$$f_1 \leq f_2 \iff f_1(t) \leq f_2(t) \quad \text{for all } t \in M.$$

In this context, the tensor product of M and L in Sup can be introduced by $M \otimes L = [M, L^{op}]^{op}$, where L^{op} is the dual lattice of L. Hence tensors in $M \otimes L$ are join-reversing maps $M \xrightarrow{f} L$ and are again ordered pointwisely.

The corresponding universal bimorphism $M \times L \xrightarrow{\otimes} M \otimes L$ has the form:

$$(t \otimes a)(s) = \begin{cases} 1, & s = 0, \\ a, & s \neq 0,\ s \leq t, \\ 0, & s \nleq t, \end{cases} \qquad s \in M.$$

Given $f \in M \otimes L$, $t \in M$ and $a \in L$, we recall the equivalence:

$$t \otimes a \leq f \iff a \leq f(t). \tag{2.1}$$

In particular, every tensor $f \in M \otimes L$ has a natural representation which can be expressed as follows:

$$f = \bigvee_{t \in M} t \otimes f(t). \tag{2.2}$$

For further details on the tensor product in Sup the reader is referred to [8] and [3, Section 2.1]. In this context we point out that meets in $M \otimes L$ are computed pointwisely, but *not* joins.

Finally, we recall the *totally below relation* on a complete lattice (cf. [2] and [4, Exercise IV.2.29]). An element $s \in M$ is totally below an element $s \in M$ (in symbols: $s \lhd t$) if for every subset S of M with $t \leq \bigvee S$ there is an $r \in S$ such that $s \leq r$. A complete lattice M is completely distributive if and only if the totally below relation is *approximating* — i.e. $t = \bigvee\{s \in M \mid s \lhd t\}$ for all $t \in M$. This implies that $\lhd$ has the *interpolation property* — i.e. if $s \lhd t$, then $s \lhd r \lhd t$ for some $r \in M$. Finally, for each $t \in M$ we write

$$\Downarrow\! t = \{s \in M \mid s \lhd t\} \qquad \text{and} \qquad \Uparrow\! t = \{s \in M \mid t \lhd s\}.$$

3 First characterizations of complete distributivity

Since in general joins in the tensor product of complete lattices cannot be computed pointwisely, the desire to give a simple description of joins leads to the following characterization of complete distributivity. The fact that complete distributivity is sufficient for this description has already appeared as Lemma 2.1.19 in [3].

Theorem 1 *Let M and L be complete lattices. Then the following assertions are equivalent*:

(1) *For every nonempty subset $\{f_i\}_{i \in I}$ of $M \otimes L$ the supremum $f = \bigvee_{i \in I} f_i$ in $M \otimes L$ is given by*:

$$f(t) = \bigwedge_{s \lhd t} \bigvee_{i \in I} f_i(s), \qquad t \in M. \tag{3.1}$$

(2) *The lattice M is completely distributive.*

Proof. (1)⇒(2): Let $t \in M \setminus \{0\}$ and put $\widehat{t} = \bigvee_{s \lhd t} s$. Then the relation $\bigvee_{s \lhd t}(s \otimes 1) = \widehat{t} \otimes 1$ holds. Now we apply (1) and obtain

$$\big(\widehat{t} \otimes 1\big)(t) = \bigwedge_{r \lhd t} \bigvee_{s \lhd t} (s \otimes 1)(r) = 1.$$

Hence $t \leq \widehat{t}$ follows — i.e. $\lhd$ is approximating, and M is completely distributive.

(2)⇒(1): Since the totally below relation on M satisfies the insertion property, the map $M \xrightarrow{\overline{f}} L$ defined by

$$\overline{f}(t) = \bigwedge_{s \lhd t} \bigvee_{i \in I} f_i(s), \qquad t \in M,$$

is join-reversing and consequently a tensor of $M \otimes L$. In order to verify (3.1) we have to show that $\overline{f}$ is the join of the family $\mathcal{F} = \{f_i \mid i \in I\}$. Since $\overline{f}(t) = \bigwedge_{s \lhd t} \bigvee_{i \in I} f_i(s) \geq \bigvee_{i \in I} f_i(t)$, it follows that $\overline{f}$ is an upper bound of $\mathcal{F}$.

On the other hand, if $g \in M \otimes L$ is an arbitrary upper bound of $\mathcal{F}$, then, since $\lhd$ is approximating in M, we notice that

$$g(t) = g\big(\bigvee_{s \lhd t} s\big) = \bigwedge_{s \lhd t} g(s) \geq \bigwedge_{s \lhd t} \bigvee_{i \in I} f_i(s) = \overline{f}(t)$$

for each $t \in M$. Hence $\overline{f}$ is the smallest upper bound of $\mathcal{F}$, and so $\overline{f} = f$. □

Corollary 1 *Let M and L be complete lattices. If M is completely distributive, then for each $\{t_i \otimes a_i\}_{i \in I} \subseteq M \otimes L$ the join $f = \bigvee_{i \in I} t_i \otimes a_i$ in $M \otimes L$ is given by*

$$f(t) = \bigwedge_{s \lhd t} \bigvee_{i \in I} (t_i \otimes a_i)(s) = \bigwedge_{s \lhd t} \big(\bigvee_{i \in I,\, s \leq t_i} a_i \big), \qquad t \in M. \tag{3.2}$$

Remark 1 Note that the formulas (3.1) and (3.2) can be slightly generalized to

$$f(t) = \bigwedge_{s \in B_t} \bigvee_{i \in I} f_i(s)$$

and

$$f(t) = \bigwedge_{s \in B_t} \bigvee_{i \in I} (t_i \otimes a_i)(s) = \bigwedge_{s \in B_t} \big(\bigvee_{i \in I,\, s \leq t_i} a_i \big)$$

for any $B_t \subseteq \Downarrow t$ such that $\bigvee B_t = t$.

Now we present an equivalent formulation of Proposition 3.6 in [6]:

Lemma 1 *Let M be a complete lattice. Then M is completely distributive if and only if for every $t \in M$ the following property holds*:

$$t \geq \bigwedge_{r \not\leq t} \big(\bigvee_{r \not\leq s} s \big). \tag{3.3}$$

Proof. If we denote $s_r = \bigvee\{s \in M \mid r \not\leq s\}$ then it follows from [6, Lemma 3.5] that:

$$t \geq \bigwedge_{r \not\leq t} \big(\bigvee_{r \not\leq s} s \big) = \bigwedge_{r \not\leq t} s_r \iff {\downarrow} t \supseteq \bigcap_{r \not\leq t} {\downarrow} s_r \iff M \setminus {\downarrow} t \subseteq \bigcup_{r \not\leq t} {\Uparrow} r.$$

Hence the assertion follows from [6, Proposition 3.6]. $\square$

Since complete distributivity is a self-dual property, we also have:

Lemma 2 *Let M be a complete lattice. Then M is completely distributive if and only if for every $t \in M$ the following property holds*:

$$t \leq \bigvee_{t \not\leq r} \big(\bigwedge_{s \not\leq r} s \big). \tag{3.4}$$

Let L be a complete lattice and $\mathsf{Up}(L)$ be the complete lattice of all upclosed subsets of L. Then $(\mathsf{Up}(L), \subseteq^{op})$ is completely distributive, and the totally below relation has the form:

$$B \lhd A \text{ and } A \neq L \iff \text{ there exists } a \notin A \text{ such that } L \setminus {\downarrow} a \subseteq B.$$

In this setting, we have the following useful relation:

$$A = \bigcap_{a \notin A} L \setminus {\downarrow} a, \qquad A \in \mathsf{Up}(L). \tag{3.5}$$

Remark 2 In what follows we shall additionally use the well known Raney's characterization of complete distributivity (cf. [9]): a lattice M is completely distributive if and only if

$$\bigvee\big(\bigcap_{i \in I} T_i\big) = \bigwedge_{i \in I} \bigvee T_i \tag{3.6}$$

for any family $\{T_i \mid i \in I\}$ of downclosed subsets of M – i.e. the formation of joins $\mathsf{Dwn}(M) \xrightarrow{\bigvee} M$ is meet-preserving where $\mathsf{Dwn}(M)$ is the complete lattice of all downclosed subsets of M.

It should be remarked that the sufficiency parts of Lemmas 1 and 2 (and thus the sufficiency part of Proposition 3.6 of [5], where the totally below relation has been used) also follows from (3.6). Indeed, since $\bigcap_{r \not\leq t} (M \setminus {\uparrow} r) = {\downarrow} t$, we have for example:

$$\bigwedge_{r \not\leq t} \bigvee_{r \not\leq s} s = \bigwedge_{r \not\leq t} \bigvee (M \setminus {\uparrow} r) = \bigvee \big(\bigcap_{r \not\leq t} (M \setminus {\uparrow} r) \big) = t.$$

4 Further characterizations of complete distributivity

In this section we give characterizations of complete distributivity based on the tensor product of complete lattices and derive some corollaries from this situation.

Let $M \times L \xrightarrow{b} \mathsf{Up}(L) \otimes M$ be given by $b(t,a) = (\uparrow a) \otimes t$ for $t \in M$ and $a \in L$. Since b is a bimorphism, it follows from the universal property of the tensor product that there exists a unique join-preserving map

$$M \otimes L \xrightarrow{\Phi} \mathsf{Up}(L) \otimes M$$

making the following diagram commutative:

$$\begin{tikzcd} M \times L \arrow[r, "\otimes"] \arrow[d, "b"'] & M \otimes L \arrow[dl, dotted, "\Phi"] \\ \mathsf{Up}(L) \otimes M & \end{tikzcd}$$

Further, if $f \in M \otimes L$, then we use its representation given in (2.2) and obtain

$$\Phi(f) = \bigvee_{t \in M} (\uparrow f(t)) \otimes t, \qquad f \in M \otimes L. \tag{4.1}$$

Since $\mathsf{Up}(L)$ is completely distributive, we invoke Corollary 1, formula (3.5), Remark 1 and conclude from (4.1) that Φ has the following explicit form:

$$\big(\Phi(f)\big)(A) = \bigwedge_{a \notin A} \bigvee f^{-1}(L \setminus \downarrow a), \qquad f \in M \otimes L,\ A \in \mathsf{Up}(L). \tag{4.2}$$

The right adjoint map $\Phi^{\vdash}$ of Φ is obviously given by:

$$\Phi^{\vdash}(\xi) = \bigvee \{ t \otimes a \in M \otimes L \mid t \leq \xi(\uparrow a)\}, \qquad \xi \in \mathsf{Up}(L) \otimes M. \tag{4.3}$$

Lemma 3 *Let M be a complete lattice. Then the following are equivalent*:

(1) *M is completely distributive.*
(2) *For each complete lattice L the relation*

$$\big(\Phi(f)\big)(A) = \bigvee f^{-1}(A) \tag{4.4}$$

holds for all $f \in M \otimes L$ and $A \in \mathsf{Up}(L)$.
(3) *Φ is injective for each complete lattice L.*

Proof. (1)$\Rightarrow$(2): Let M be a completely distributive lattice, L be a complete lattice, $f \in M \otimes L$ and $A \in \mathsf{Up}(L)$. Since f is order-reversing, $f^{-1}(L \setminus \downarrow a)$ is a downclosed set for all $a \in L$. By using (4.2), (3.6), and (3.5), we obtain:

$$\begin{aligned} \big(\Phi(f)\big)(A) &= \bigwedge_{a \notin A} \bigvee f^{-1}(L \setminus \downarrow a)) \\ &= \bigvee \big(\bigcap_{a \notin A} f^{-1}(L \setminus \downarrow a)\big) \\ &= \bigvee f^{-1}\big(\bigcap_{a \notin A} L \setminus \downarrow a)\big) \\ &= \bigvee f^{-1}(A). \end{aligned}$$

(2)⇒(3): Let L be a complete lattice, $f \in M \otimes L$ and $a \in L$. Since f is join-reversing we have that $t \leq \bigvee\{s \in M \mid a \leq f(s)\}$ if and only if $a \leq f(t)$. Now we refer to (4.3) and apply (4.4). Then we obtain:

$$\Phi^{\vdash}(\Phi(f)) = \bigvee\{t \otimes a \mid t \leq \bigvee f^{-1}(\uparrow a)\} = \bigvee\{t \otimes a \mid a \leq f(t)\} = f.$$

Hence Φ is injective.

(3)⇒(1): We choose $L = M^{op}$ and assume that Φ is injective. Hence for the identity map $1_M \in M \otimes M^{op}$ the relation $\Phi^{\vdash}(\Phi(1_M)) = 1_M$ holds. Referring to (4.3) and (2.1) we obtain $t \leq^{op} \Phi(1_M)(\uparrow^{op} t)$ for all $t \in M$ — i.e.

$$\big(\Phi(1_M)\big)(\uparrow^{op} t) = \bigwedge_{r \not\leq t} \big(\bigvee_{r \not\leq s} s\big) \leq t, \qquad t \in M.$$

Hence M is completely distributive by Lemma 1. □

Notation Let M and L be complete lattices. We denote by

$$\langle \mathsf{Up}(L), M\rangle$$

the partially ordered set of all join- and meet-reversing maps from $\mathsf{Up}(L)$ to M.

Theorem 2 *Let M be a complete lattice. Then M is completely distributive if and only if Φ determines an order isomorphism between $M \otimes L$ and $\langle \mathsf{Up}(L), M\rangle$ for every complete lattice L.*

Proof. The sufficiency of the condition follows from the implication (3)⇒(1) of Lemma 3. On the other hand, if M is completely distributive then, as an immediate corollary of (4.4) we obtain that

$$\big(\Phi(f)\big)\big(\bigwedge_{i \in I} A_i\big) = \bigvee f^{-1}\big(\bigcup_{i \in I} A_i\big) = \bigvee_{i \in I} \big(\Phi(f)\big)(A_i)$$

holds for each $f \in M \otimes L$ and $\{A_i\}_{i \in I} \subseteq \mathsf{Up}(L)$ — i.e. $\Phi(f)$ is meet-reversing for all $f \in M \otimes L$. In order to show that the range of Φ coincides with the set of all meet-reversing tensors of $\mathsf{Up}(L) \otimes M$, we choose $\varphi \in \mathsf{Up}(L) \otimes M$ and observe that φ induces a tensor $f \in M \otimes L$ as follows:

$$f(t) = \bigvee\{a \in L \mid t \leq \varphi(\uparrow a)\}, \qquad t \in M.$$

Since φ is join-reversing, the equivalence $a \leq f(t) \iff t \leq \varphi(\uparrow a)$ holds for all $t \in M$ and $a \in L$. Now, we apply again Lemma 3 and obtain:

$$\big(\Phi(f)\big)(\uparrow a) = \bigvee\{t \in M \mid a \leq f(t)\} = \varphi(\uparrow a).$$

Finally, if we assume that φ is meet-reversing, then $\Phi(f)$ coincides with φ. □

In the following considerations we give a characterization of elements of $\langle \mathsf{Up}(L), M\rangle$ as continuous maps. First we add some terminology. An upclosed subset P of L is said to be *prime* if the following implication holds for all $a, b \in L$:

$$a \vee b \in P \quad \text{implies} \quad a \in P \text{ or } b \in P.$$

The set $\mathsf{PUp}(L)$ of all prime upclosed subsets of L is a complete lattice with respect to the partial order inherited from $\mathsf{Up}(L)$. Since every upclosed subset A of L has the form $A = \bigcap_{a \notin A} L \setminus \downarrow a$, the tensor products $\mathsf{Up}(L) \otimes M$ and $\mathsf{PUp}(L) \otimes M$ are order isomorphic. Further, we consider the interval topology on $\mathsf{Up}(L)$. Since for all $A \in L$ the relation

$$\{A \in \mathsf{Up}(L) \mid \uparrow a \subseteq A\} = \mathsf{Up}(L) \setminus \{A \in \mathsf{Up}(L) \mid A \subseteq L \setminus \downarrow a\}$$

holds, it is easily seen that $\mathsf{Up}(L)$ is a totally disconnected, compact Hausdorff space, and $\mathsf{PUp}(L)$ is a closed subset of $\mathsf{Up}(L)$. Hence we view $\mathsf{PUp}(L)$ as a compact Hausdorff space w.r.t. the relative topology induced by $\mathsf{Up}(L)$ on $\mathsf{PUp}(L)$.

Corollary 2 *Let M be a completely distributive lattice and L be an arbitrary complete lattice. Then every tensor $f \in M \otimes L$ can be identified with a continuous map $\mathsf{PUp}(L) \xrightarrow{\varphi_f} M$ defined by:*

$$\varphi_f(A) = \bigvee f^{-1}(A), \qquad A \in \mathsf{PUp}(L) \tag{4.5}$$

where M is provided with its interval topology.

Proof. We use the order isomorphism between $\mathsf{Up}(L) \otimes M$ and $\mathsf{PUp}(L) \otimes M$ and conclude from Lemma 3 and Theorem 2 that every tensor $f \in M \otimes L$ can be identified with a map $\mathsf{PUp}(L) \xrightarrow{\varphi_f} M$ determined by (4.5). We have only to show that φ_f is continuous w.r.t. the interval topology on M. In fact the following relations hold:

$$\varphi_f^{-1}(M \setminus \downarrow t) = \bigcup_{r \not\leq t} \{A \in \mathsf{PUp}(L) \mid f(r) \in A\}, \qquad t \in M, \tag{4.6}$$

$$\varphi_f^{-1}(M \setminus \uparrow t) = \bigcup_{s \lhd t} \{A \in \mathsf{PUp}(L) \mid f(s) \notin A\}, \qquad t \in M, \tag{4.7}$$

where $\lhd$ is the totally below relation in M. □

As an illustration of the previous results we present a measure-theoretical description of tensors in $[0, +\infty]^{op} \otimes L$.

Example 1 Let $[0, +\infty]^{op}$ be the nonnegative extended real numbers provided with the dual ordering. Further, let L be a complete lattice and $L \xrightarrow{\nu} [0, 1]$ be a map satisfying the following conditions:

$$\nu(0) = 0, \quad \nu(1) = 1, \quad \nu(a_1 \vee a_2) = \max(\nu(a_1), \nu(a_2)), \quad a_1, a_2 \in L. \tag{4.8}$$

Obviously, ν induces a Baire measurable map $[0,1) \xrightarrow{\Upsilon} \mathsf{PUp}(L)$ by:

$$\Upsilon(r) = \{a \in L \mid r < \nu(a)\}, \qquad r \in [0,1).$$

Hence there exists a unique regular Borel probability measure μ_ν on $\mathsf{PUp}(L)$ satisfying the property

$$\mu_\nu\big(\{A \in \mathsf{PUp}(L) \mid a \in A\}\big) = \nu(a)$$

for all $a \in L$ where we have used the Lebesgue measure on $[0,1)$.

Referring to Corollary 2 it is easily seen that each $f \in [0,+\infty]^{op} \otimes L$ can be identified with a nonnegative extended real-valued random variable φ_f. In fact, if we additionally assume the σ-smoothness of ν — i.e.

$$a_{n+1} \le a_n, \qquad \nu\big(\bigwedge_{n\in\mathbb{N}} a_n\big) = \inf_{n\in\mathbb{N}} \nu(a_n),$$

then an application of (4.7) and (4.8) shows that φ_f satisfies the following relation for all $t \in [0,+\infty]$:

$$\mu_\nu\big(\varphi_f^{-1}(\uparrow t)\big) = \mu_\nu\big(\bigcap_{t<s}\{A \in \mathsf{PUp}(L) \mid f(s) \in A\}\big) = \inf_{t<s} \nu(f(s)) = \nu(f(t)).$$

If L coincides with the real unit interval $[0,1]$ provided with the identity $\nu = 1_{[0,1]}$, then the previous observation is the well-known fact that every distribution function can be viewed as a distribution of an appropriate random variable (cf. [10, Exercise A 1.6, p. 221]). In this sense tensors of $[0,+\infty]^{op} \otimes L$ are abstract L-valued distributions on $[0,+\infty]$.

In the previous example all tensors are meet-preserving maps from $[0,+\infty]$ to a complete lattice L. Sometimes, especially in probability theory, it is desirable to work with join-preserving maps. This is the reason why we now present an order isomorphism between $M^{op} \otimes L$ and $[M, L]$ in the case of a completely distributive lattice M.

Let us begin with arbitrary complete lattices M and L and recall an adjoint pair of isotone maps $M^{op} \otimes L \xrightarrow{\Gamma} [M, L]$ and $[M, L] \xrightarrow{\Delta} M^{op} \otimes L$ defined as follows:

$$\big(\Gamma(f)\big)(t) = \bigvee_{t \not\le s} f(s) \quad \text{ and } \quad \big(\Delta(g)\big)(t) = \bigwedge_{s \not\le t} g(s)$$

for each $t \in M$, $f \in M^{op} \otimes L$ and $g \in [M, L]$. Obviously, $\Gamma(f)$ is join-preserving — i.e. $\Gamma(f) \in [M, L]$ and $\Delta(g)$ is meet-preserving — i.e. $\Delta(g) \in M^{op} \otimes L$.

Lemma 4 *Let M and L be complete lattices. The map Γ is left adjoint to Δ.*

Proof. It follows immediately from the definition of Γ and Δ that the following relations hold:

$$f(t) \le \bigwedge_{s \not\le t}\big(\bigvee_{s \not\le r} f(r)\big) \quad \text{ and } \quad \bigvee_{t \not\le s}\big(\bigwedge_{r \not\le s} g(r)\big) \le g(t)$$

for each $t \in M$, $f \in M^{op} \otimes L$ and $g \in [M, L]$. Hence the assertion follows. $\square$

Proposition 1 *Let M be a complete lattice. Then the map*

$$M^{op} \otimes L \xrightarrow{\Gamma} [M, L]$$

is injective for every complete lattice L if and only if M is completely distributive.

Proof. In order to verify the necessity of the condition we choose $L = M$ and conclude from $\Delta(\Gamma(1_M)) = 1_M$ that for every $t \in M$ the relation

$$\bigwedge_{s \not\leq t} \Big(\bigvee_{s \not\leq r} r \Big) = t$$

holds. Hence the complete distributivity of M follows from Lemma 1.

On the other hand, let us assume that M is completely distributive. We fix $f \in M^{op} \otimes L$ and choose an element $t \in M$. Then we obtain:

$$\big(\Delta(\Gamma(f))\big)(t) = \bigwedge_{s \not\leq t} \Big(\bigvee_{s \not\leq r} f(r) \Big).$$

Since $M \xrightarrow{f} L$ is meet-preserving, we apply again Lemma 1 and conclude from the previous relation that

$$f(t) \leq \big(\Delta(\Gamma(f)\big)(t) \leq \bigwedge_{s \not\leq t} f\Big(\bigvee_{s \not\leq r} r \Big) = f\Big(\bigwedge_{s \not\leq t} \Big(\bigvee_{s \not\leq r} r \Big)\Big) = f(t).$$

Hence $\Delta(\Gamma(f)) = f$ follows — i.e. Γ is injective. □

Corollary 3 *If M is completely distributive, then $[M, L] \xrightarrow{\Delta} M^{op} \otimes L$ is surjective.*

Proof. The assertion follows from Proposition 1 and the adjunction $\Gamma \dashv \Delta$. □

We can summarize the previous result as follows.

Fact 1 *If M is completely distributive, then $M^{op} \otimes L \xrightarrow{\Gamma} [M, L]$ determined by*

$$\big(\Gamma(f)\big)(t) = \bigvee_{t \not\leq s} f(s), \qquad t \in M,\ f \in M^{op} \otimes L$$

is an order isomorphism and its inverse map $[M, L] \xrightarrow{\Delta} M^{op} \otimes L$ has the following form:

$$\big(\Delta(g)\big)(t) = \bigwedge_{s \not\leq t} g(s), \qquad t \in M,\ g \in [M, L].$$

5 Characterization of the tensor product of quantales with one completely distributive factor

Since quantales are semigroups in Sup and Sup is a symmetric monoidal category, the tensor product of quantales exists. Let us fix quantales $(M,\odot)$ and $(L,*)$. Then the multiplication $\star$ of the tensor product $M\otimes L$ is uniquely determined on elementary tensor as follows (cf. [3, Section 2.3]):

$$(t_1\otimes a_1)\star(t_2\otimes a_2)=(t_1\odot t_2)\otimes(a_1*a_2),\qquad t_1,t_2\in M,\ a_1,a_2\in L.$$

Hence, if $f_1,f_2\in M\otimes L$, then $f_1\star f_2$ is given by:

$$\begin{aligned}f_1\star f_2&=\bigvee_{s_1,s_2\in M}(s_1\otimes f_1(s_1))\star(s_2\otimes f_2(s_2))\\&=\bigvee_{s_1,s_2\in M}(s_1\odot s_2)\otimes\big(f_1(s_1)\star f_2(s_2)\big).\end{aligned}$$

Now we assume that M is completely distributive. The advantage of this assumption is that we can give an explicit description of $f_1\star f_2$. Referring to Corollary 1 or to formula (5.2) in [5, Corollary 5.2] the following relation holds:

$$(f_1\star f_2)(t)=\bigwedge_{s\lhd t}\big(\bigvee_{s\le s_1\odot s_2}f_1(s_1)*f_2(s_2)\big),\quad t\in M,\ f_1,f_2\in M\otimes L.\quad(5.1)$$

A first application of this observation is the following remark.

Remark 3 After Fact 1 we have now a quantale multiplication on $[M,L]$ induced by the order isomorphism $[M,L]\xrightarrow{\Delta}M^{op}\otimes L$. Given $g_1,g_2\in[M,L]$ and $t\in M$, then

$$(g_1\star g_2)(t):=\big(\Gamma\big(\Delta(g_1)\star\Delta(g_2)\big)\big)(t).$$

In the special case $(M,\odot)=([0,+\infty]^{op},+)$ we have for $g_1,g_2\in[[0,+\infty],L]$ and $t\in[0,+\infty]$:

$$\begin{aligned}(g_1\star g_2)(t)&=\bigvee_{u<t}\Big(\bigwedge_{s>u}\big(\bigvee_{s\ge s_1+s_2}\big(\Delta(g_1)\big)(s_1)*\big(\Delta(g_2)\big)(s_2)\big)\Big)\\&=\bigvee_{u<t}\Big(\bigwedge_{s>u}\big(\bigvee_{s\ge s_1+s_2}\big(\bigwedge_{r_1>s_1}g_1(r_1)\big)*\big(\bigwedge_{r_2>s_2}g_2(r_2)\big)\big)\Big)\\&=\bigvee_{u<t}\Big(\bigwedge_{s>u}\big(\bigvee_{s=s_1+s_2}g_1(s_1)*g_2(s_2)\big)\Big)\\&=\bigvee_{t=s_1+s_2}g_1(s_1)*g_2(s_2)\big)\end{aligned}$$

The previous relation can be understood as an L-valued generalization of the Fact in [5, Section 6] for any complete lattice L.

Further, we recall the (M, L)-component $M \otimes L \xrightarrow{c_{ML}} L \otimes M$ of the symmetry in Sup:

$$\big(c_{ML}(f)\big)(a) = \bigvee\{t \in M \mid a \leq f(t)\}, \qquad a \in L,\ f \in M \otimes L.$$

Then $M \otimes L \xrightarrow{c_{ML}} L \otimes M$ is a quantale isomorphism. The next theorem shows which impact formula (5.1) has on the description of the multiplication in $L \otimes M$.

Theorem 3 *Let M be completely distributive. Then the product of $g_1, g_2 \in L \otimes M$ with respect to the multiplication in $L \otimes M$ has the following form:*

$$(g_1 \star g_2)(a) = \bigwedge_{a \not\leq b} \Big(\bigvee_{c_1 * c_2 \not\leq b} g_1(c_1) \odot g_2(c_2) \Big), \qquad a \in L,\ g_1, g_2 \in L \otimes M.$$

Proof. If $g_1, g_2 \in L \otimes M$, then we define $\bar{g}_1 = c_{LM}(g_1)$ and $\bar{g}_2 = c_{LM}(g_2)$. Then

$$(g_1 \star g_2)(a) = \big(c_{ML}(\bar{g}_1 \star \bar{g}_2)\big)(a) = \bigvee\{t \in M \mid a \leq (\bar{g}_1 \star \bar{g}_2)(t)\}$$

for each $a \in L$.

Now we verify the assertion of the theorem and fix $t \in M$ and $a \in L$ such that $a \leq (\bar{g}_1 \star \bar{g}_2)(t)$. Then we infer from formula (5.1) that

$$a \leq (\bar{g}_1 \star \bar{g}_2)(t) = \bigwedge_{s \lhd t} \Big(\bigvee_{s \leq s_1 \odot s_2} \bar{g}_1(s_1) * \bar{g}_2(s_2) \Big).$$

Hence for every $s \lhd t$ and for every $a \not\leq b$ there exist $s_1, s_2 \in M$ such that $s \leq s_1 \odot s_2$ and $\bar{g}_1(s_1) * \bar{g}_2(s_2) \not\leq b$. Now we put $c_1 = \bar{g}_1(s_1)$ and $c_2 = \bar{g}_2(s_2)$ and obtain

$$s \leq s_1 \odot s_2 \leq g_1(c_1) \odot g_2(c_2) \quad \text{and} \quad c_1 * c_2 \not\leq b.$$

Since $\lhd$ is approximating, the relation

$$t = \bigvee_{s \lhd t} s \leq \Big(\bigvee_{c_1 * c_2 \not\leq b} g_1(c_1) \odot g_2(c_2) \Big)$$

follows. Hence

$$(g_1 \star g_2)(a) = \bigvee\{t \in M \mid a \leq (\bar{g}_1 \star \bar{g}_2)(t)\} \leq \bigwedge_{a \not\leq b} \Big(\bigvee_{c_1 * c_2 \not\leq b} g_1(c_1) \odot g_2(c_2) \Big).$$

On the other hand, let $t_0 = \bigwedge_{a \not\leq b} \big(\bigvee_{c_1 * c_2 \not\leq b} g_1(c_1) \odot g_2(c_2) \big)$ and choose $s \in M$ with $s \lhd t_0$. Then for every $b \in L$ with $a \not\leq b$ there exist $c_1, c_2 \in L$ such that

$$c_1 * c_2 \not\leq b \quad \text{and} \quad s \leq g_1(c_1) \odot g_2(c_2).$$

Now we put $s_1 = g_1(c_1)$ and $s_2 = g_2(c_2)$ and obtain

$$s \leq s_1 \odot s_2, \quad c_1 \leq \bar{g}_1(s_1), \quad c_2 \leq \bar{g}_2(s_2) \quad \text{and} \quad \bar{g}_1(s_1) * \bar{g}_2(s_2) \not\leq b.$$

Hence $\bigvee_{s \leq s_1 \odot s_2} \bar{g}_1(s_1) * \bar{g}_2(s_2) \not\leq b$ for all $b \in L$ with $a \not\leq b$. This observation implies that

$$a \leq \bigvee_{s \leq s_1 \odot s_2} \bar{g}_1(s_1) * \bar{g}_2(s_2).$$

Since $s \lhd t_0$ is arbitrary, we obtain the following relation:

$$a \leq \bigwedge_{s \lhd t_0} \Big(\bigvee_{s \leq s_1 \odot s_2} \bar{g}_1(s_1) * \bar{g}_2(s_2) \Big).$$

With regard to formula (5.1) this means $a \leq (\bar{g}_1 \star \bar{g}_2)(t_0)$. Hence we conclude:

$$\bigwedge_{a \not\leq b} \Big(\bigvee_{c_1 * c_2 \not\leq b} g_1(c_1) \odot g_2(c_2) \Big) = t_0 \leq (g_1 \star g_2)(a). \qquad \square$$

Remark 4 (1) Let M be completely distributive. Since $L \otimes M^{op} = [L, M]^{op}$, the composite of $[M, L] \xrightarrow{\Delta} M^{op} \otimes L$ and $c_{M^{op}L}$ leads to an order reversing isomorphism $[M, L] \xrightarrow{\Theta} [L, M]$ determined by:

$$\big(\Theta(g)\big)(a) = \bigwedge\{t \in M \mid a \leq \big(\Delta(g)\big)(t)\} = \bigwedge\big\{t \in M \mid a \leq \bigwedge_{s \not\leq t} g(s)\big\}$$

for each $a \in L$ and $g \in [M, L]$. In the particular case $M = [0, +\infty]$ the order reversing isomorphism is given by:

$$\big(\Theta(g)\big)(a) = \inf\{t \in [0, +\infty] \mid a \leq g(t)\} = \sup\{t \in [0, +\infty] \mid a \not\leq g(t)\}$$

for each $a \in L$ and $g \in [[0, +\infty], L]$.

(2) If we replace the arbitrary complete lattice L by the real unit interval $[0, 1]$, then it is easily seen that the map Θ is the generalization of the correspondence $g \longmapsto g^{\frown}$ given by B. Schweizer's and A. Sklar's construction of the left-continuous quasi-inverse function $g^{\frown}$ from a given probability distribution function g (cf. [11, pages 50–51]). It is well-known that this correspondence plays a significant role in the study of triangle functions in the theory of probabilistic metric spaces. In this context the formula (7.7.8) in [11, page 116] follows immediately from our Theorem 3. In fact, since Θ is an order-reversing semigroup isomorphism, we obtain the following relation:

$$\big(\Theta(g_1 \star g_2)\big)(a) = \sup_{a \not\leq b}\Big(\inf_{c_1 * c_2 \not\leq b} \big(\Theta(g_1)\big)(c_1) + \big(\Theta(g_2)\big)(c_2)\Big), \qquad a \in L,$$

where $(L, *)$ is an arbitrary quantale and $g_1, g_2 \in [[0, +\infty], L]$.

(3) If we now replace the quantale $(L, *)$ by the real unit interval provided with a continuous t-norm — i.e. $([0, 1], *)$ is a continuous quantale (cf. [3, Subsection 2.3.4]), then finally the relation

$$\big(\Theta(g_1 \star g_2)\big)(a) = \sup_{b < a}\Big(\inf_{c_1 * c_2 = b} \big(\Theta(g_1)\big)(c_1) + \big(\Theta(g_2)\big)(c_2)\Big), \qquad a \in [0, 1].$$

holds. Hence as a by-product of these investigations we obtain that the characterization of the tensor product of quantales with one completely distributive factor reveals the fundamental principles hidden in B. Schweizer's and

A. Sklar's construction of the left-continuous quasi-inverse function — these are the components of the symmetry of the tensor product in Sup corresponding to the pair $([0,+\infty]^{op}, [0,1])$ and the quantale isomorphism from $[[0,+\infty],[0,1]]$ to the tensor product $[0,+\infty]^{op} \otimes [0,1]$.

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