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Cohomology of p-groups of nilpotency class smaller than p

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Abstract. Let p be a prime number, let d be an integer and let G be a d-generated finite p-group of nilpotency class smaller than p. Then the number of possible isomorphism types for the mod p cohomology algebra $H^*(G;\mathbb{F}_p)$ is bounded in terms of p and d.

1 Introduction

Let p be a prime number and let G be a finite p-group of order p^n and nilpotency class c, then G has coclass $m = n - c$. In 2005, J. F. Carlson proved that for a fixed integer m, there are only finitely many possible isomorphism types for the mod 2 cohomology algebra of 2-groups of coclass m (see [1, Theorem 5.1]). In the same paper, Carlson conjectures that the analogous result should hold for the p odd case, that is, he conjectures that there are finitely many isomorphism types of cohomology algebras for all p-groups of fixed coclass. This result has been partially proved first by Eick and Green in [4] and later by the authors of this manuscript in [3]. In the former paper, Eick and Green prove that there are finitely many Quillen categories of p-groups of fixed coclass m. In the latter paper, Díaz Ramos, Garaialde Ocaña and González-Sánchez prove Carlson's conjecture for the *non-twisted* p-groups of coclass m [3, Theorem 7.1]. For instance, all the 2-groups of fixed coclass are non-twisted.

The general case of the above conjecture is still open. One of the main differences between the $p = 2$ case and the p odd case is that in the even case, 2-groups of coclass m have a “large” abelian subsection, while in the p odd case, such p-groups have a “large” subsection of nilpotency class 2.

A dual problem to Carlson's conjecture would be to study the number of possible isomorphism types for the mod p cohomology algebras $H^*(G;\mathbb{F}_p)$ of d-generated finite p-groups G of nilpotency class c. In this note we prove that whenever the nilpotency class c of a d-generated finite p-group G is smaller than p, then the number of possible isomorphism types of cohomology algebras $H^*(G;\mathbb{F}_p)$ is bounded in terms of p and d.

Main Theorem. *The number of possible isomorphism types for the mod p-cohomology algebra of a d-generated finite p-group of nilpotency class smaller than p is bounded by a function depending only on p and d.*

The key ingredients for proving the above result are the Lazard correspondence, a structure result for finite nilpotent $\mathbb{Z}_p$-Lie algebras, the description of the mod p cohomology algebra of powerful p-central p-groups with the Ω-extension property given by T. S. Weigel (see [12]) and some refinements on Carlson's counting arguments from [1].

Notation. Let G be a group, G^{p^k} denotes the subgroup generated by the p^k powers of G and $\Omega_k(G)$ denotes the subgroup generated by the elements of order at most p^k. The *reduced* mod p cohomology algebra $H^*(G;\mathbb{F}_p)_{\mathrm{red}}$ is the quotient $H^*(G;\mathbb{F}_p)/\mathrm{nil}(H^*(G;\mathbb{F}_p))$, where $\mathrm{nil}(H^*(G;\mathbb{F}_p))$ is the ideal of all nilpotent elements in the mod p cohomology algebra. We will use $[\cdot,\cdot]$ to denote both the Lie bracket of a Lie algebra and the group commutator. If D is either a group or a Lie algebra and I is either a normal subgroup or an ideal of D respectively, then we denote

$$[I,{}_c D] = [I, \overbrace{D,\dots,D}^{c}].$$

We say that a function k is $(a,b,\dots)$-bounded if there exists a function f depending on $(a,b,\dots)$ such that $k \le f(a,b\dots)$. The rank of a p-group G is the sectional rank, that is,

$$\mathrm{rk}(G) = \max\{d(H) \mid H \le G\},$$

where $d(H)$ is the minimal number of generators of H.

2 Preliminaries

2.1 Powerful p-central groups and $\mathbb{Z}_p$-Lie algebras.

Following [12], we define powerful p-central groups and $\mathbb{Z}_p$-Lie algebras with the Ω-extension property (ΩEP for short). We also recall a result about the cohomology algebra of such p-groups. Recall that for a p-group G, we denote

$$\Omega_1(G) = \langle g \in G \mid g^p = 1\rangle,$$

and for a $\mathbb{Z}_p$-Lie algebra L, we denote

$$\Omega_1(L) = \{a \in L \mid p.a = 0\}.$$

Definition 2.1. Let p be an odd prime. Let G be a p-group and let L be a finite $\mathbb{Z}_p$-Lie algebra. Then:

(i) G is *powerful* if $[G,G] \subset G^p$ and L is *powerful* if $[L,L] \subset pL$.

(ii) G is *p-central* if its elements of order p are contained in the center of G and L is *p-central* if its elements of order p are contained in the center of L.

(iii) G has the *Ω-extension property* (ΩEP for short) if there exists a p-central group H such that $G = H/\Omega_1(H)$ and L has the *Ω-extension property* if there exists a p-central Lie algebra A such that $L = A/\Omega_1(A)$.

An easy computation shows that if G (or L) has the ΩEP, then G (or L) is p-central. We finish this subsection by describing the mod p cohomology algebra of a powerful p-central p-group with ΩEP.

Theorem 2.2. *Let p be an odd prime, let G be a powerful p-central p-group with the* ΩEP *and let d denote the $\mathbb{F}_p$-rank of $\Omega_1(G)$. Then:*

(a) *$H^*(G;\mathbb{F}_p) \cong \Lambda(y_1,\ldots,y_d) \otimes \mathbb{F}_p[x_1,\ldots,x_d]$ with $|y_i| = 1$ and $|x_i| = 2$,*

(b) *the reduced restriction map $j_{\mathrm{red}} : H^*(G;\mathbb{F}_p)_{\mathrm{red}} \to H^*(\Omega_1(G);\mathbb{F}_p)_{\mathrm{red}}$ is an isomorphism.*

Proof. See [12, Theorem 2.1 and Corollary 4.2] □

2.2 The Lazard correspondence for finitely generated nilpotent pro-p groups and $\mathbb{Z}_p$-Lie algebras

Let Lie_p and Gr_p denote the category of finitely generated $\mathbb{Z}_p$-Lie algebras of nilpotency class smaller than p and the category of finitely generated pro-p groups of nilpotency class smaller than p, respectively. Denote by Top the category of topological spaces and by $\mathbf{for} : \mathrm{Lie}_p \to \mathrm{Top}$ and $\mathbf{For} : \mathrm{Gr}_p \to \mathrm{Top}$ the fully faithful forgetful functors which forget the algebraic structure.

Theorem 2.3 (Lazard correspondence [10]). *There exist isomorphisms of categories, inverse to each other,*

$$\mathbf{exp} : \mathrm{Lie}_p \to \mathrm{Gr}_p \quad \text{and} \quad \mathbf{log} : \mathrm{Gr}_p \to \mathrm{Lie}_p$$

such that

$$\mathbf{for} \circ \mathbf{log} = \mathbf{For} \quad \text{and} \quad \mathbf{For} \circ \mathbf{exp} = \mathbf{for}.$$

Furthermore, if $G \in \mathrm{Gr}_p$ and K is a subgroup of G, the following statements hold:

(a) *K is a normal subgroup of G if and only if $\mathbf{log}(K)$ is an ideal in $\mathbf{log}(G)$. Moreover, $\mathbf{log}(G/K) = \mathbf{log}(G)/\mathbf{log}(K)$,*

(b) *$\mathbf{log}(\Omega_1(G)) = \Omega_1(\mathbf{log}(G))$ and $\mathbf{log}(G^p) = p\,\mathbf{log}(G)$,*

(c) *Nilpotency class of G = nilpotency class of $\mathbf{log}(G)$,*

(d) *G is a powerful p-group if and only if $\mathbf{log}(G)$ is a powerful $\mathbb{Z}_p$-Lie algebra,*

(e) *G is a p-central group if and only if $\mathbf{log}(G)$ is a p-central $\mathbb{Z}_p$-Lie algebra,*

(f) *A subset X of K generates K (topologically) if and only if X generates $\mathbf{log}(K)$ as a $\mathbb{Z}_p$-Lie algebra. In particular, the number of generators of K as a topological group coincides with the number of generators of $\mathbf{log}(K)$ as a $\mathbb{Z}_p$-Lie algebra.*

Proof. The existence of the functors **exp** and **log** was first discovered by M. Lazard in [10]. Explicit formulae can be found in [2]. One can also find the Lazard correspondence in [9, Theorem 10.13 and page 124]. More precisely: (a) follows from [9, Theorem 10.13 (b)], (b) follows from the explicit Baker–Campbell–Hausdorff formula [9, Lemma 9.15], (c) follows from [9, Theorem 10.13 (d)], (d) and (e) follow from comparing (b) of this theorem and [9, Theorem 10.13 (c)]. Finally, (f) follows from the Baker–Campbell–Hausdorff formulae. □

We finish this subsection by proving a result on finite p-groups of nilpotency class smaller than p.

Proposition 2.4. *Let p be a prime number and G a finite p-group of nilpotency class smaller than p. Then G^p is a powerful p-central p-group with the* ΩEP.

Proof. Let G be a finite p-group of nilpotency class $m < p$. By the Lazard correspondence, $G = \mathbf{exp}(L)$, where $(L, +, [\cdot,\cdot])$ is a $\mathbb{Z}_p$-Lie algebra of nilpotency class m.

The $\mathbb{Z}_p$-Lie algebra $(pL, +, [\cdot,\cdot])$ satisfies the following properties:

$$[pL, pL] = p[pL, L] \leq p(pL)$$

and

$$[\Omega_1(pL), pL] = p[\Omega_1(pL), L] = 0.$$

That is, pL is a powerful p-central $\mathbb{Z}_p$-Lie algebra of nilpotency class at most m (see Definition 2.1). By Theorem 2.3 (b), $G^p = \mathbf{exp}(pL)$ is a powerful p-central p-group of nilpotency class at most m.

It remains to show that G^p has the ΩEP. Consider $(L, +)$ as a d-generated abelian p-group for some positive integer d. Then L is isomorphic to the quotient of a free $\mathbb{Z}_p$-module M with d generators by a $\mathbb{Z}_p$-submodule I with $I \subseteq pM$. The Lie bracket $[\cdot,\cdot] \in \mathrm{Hom}(L \otimes_{\mathbb{Z}_p} L, L)$ is an antisymmetric bilinear form and it can be uniquely lifted to the bilinear form $\{\cdot,\cdot\} \in \mathrm{Hom}(M \otimes_{\mathbb{Z}_p} M, M)$. Although this bilinear form $\{\cdot,\cdot\}$ may fail to be a Lie bracket in M, for all $x, y, z \in M$, we have that

$$\{x, y, z\} + \{z, x, y\} + \{y, z, x\} \in I$$

holds, since $[\cdot,\cdot]$ satisfies the Jacobi identity in $L = M/I$. In particular, the property

$$\{px, py, pz\} + \{pz, px, py\} + \{py, pz, px\} \in p^3 I,$$

shows that the bilinear form $\{\cdot,\cdot\}$ is a Lie bracket in the quotient pM/pI. Moreover, the nilpotency class of $(pM/pI, +, \{\cdot,\cdot\})$ is at most m as

$$\{\overbrace{pM, \dots, pM}^{m+1}\} = p^{m+1}\{\overbrace{M, \dots, M}^{m+1}\} \leq p^{m+1} I.$$

As before, the properties

$$\{pM, pM\} \leq pM \quad \text{and} \quad \{\Omega_1(pM), pM\} = 0$$

show that $(pM/pI, +\{\cdot,\cdot\})$ is a powerful p-central $\mathbb{Z}_p$-Lie algebra. Notice that $\Omega_1(pM/pI) = I/pI$ and by the third isomorphism theorem

$$pL = p(M/I) = pM/I \cong \frac{pM/pI}{I/pI} = \frac{pM/pI}{\Omega_1(pM/pI)}.$$

Finally, by Theorem 2.3, we have that $\mathbf{exp}(pM/pI)$ is p-central and, as

$$\begin{aligned} G^p &= \mathbf{exp}(pL) \\ &= \mathbf{exp}\left(\frac{pM/pI}{\Omega_1(pM/pI)}\right) \\ &= \frac{\mathbf{exp}(pM/pI)}{\mathbf{exp}(\Omega_1(pM/pI))} \\ &= \frac{\mathbf{exp}(pM/pI)}{\Omega_1(\mathbf{exp}(pM/pI))}, \end{aligned}$$

we conclude that G^p is a powerful p-central p-group with the ΩEP. □

3 Finitely generated nilpotent $\mathbb{Z}_p$-Lie algebras

We start this section by giving an easy result that will be used throughout the section.

Lemma 3.1. *Let L be a $\mathbb{Z}_p$-Lie algebra and let I be an ideal in L. Then, for all $i \geq 1$,*

$$[\gamma_i(L), I] \subseteq [I,_i L],$$

where $\gamma_i(L)$ denotes the lower central series of L. In particular, for all $i, j \geq 1$, we have that $[\gamma_i(L), \gamma_j(L)] \subseteq \gamma_{i+j}(L)$.

Proof. We shall prove the result by induction on i. If $i = 1$, then the statement clearly holds. Suppose that the above inclusion holds for all $i < k$. Then

$$\begin{aligned}[\gamma_k(L), I] &= [[\gamma_{k-1}(L), L], I] \\ &\subseteq [[I, \gamma_{k-1}(L)], L] + [[L, I], \gamma_{k-1}(L)] \\ &\subseteq [[I, {}_{k-1}L], L] + [[L, I], {}_{k-1}L] = [I, {}_k L],\end{aligned}$$

where the first inclusion holds by the Jacobi identity and the second one uses the induction hypothesis. □

Let p be a prime number, let c, d be positive integers, let X be a set with d elements and let $L_c(X)$ denote the free $\mathbb{Z}_p$-Lie algebra over X of nilpotency class c. Let $\widetilde{L}_c(X) = L_c(X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ denote the free $\mathbb{Q}_p$-Lie algebra of nilpotency class c. We can assume that $L_c(X) \subseteq \tilde{L}_c(X)$. We define a $\mathbb{Z}_p$-Lie algebra,

$$\begin{aligned}\widehat{L}_c(X) := L_c(X) + \frac{1}{p}\gamma_2(L_c(X)) + \frac{1}{p^2}\gamma_3(L_c(X)) + \cdots + \frac{1}{p^{c-1}}\gamma_c(L_c(X)) \\ \subseteq \widetilde{L}_c(X).\end{aligned}$$

Lemma 3.2. *Let $\widehat{L}_c(X)$ be as defined above. Then*

(a) *$\widehat{L}_c(X)$ is a finitely generated powerful $\mathbb{Z}_p$-Lie algebra of nilpotency class c.*

(b) *The rank of $\widehat{L}_c(X)$ is (c, d)-bounded.*

(c) *The index $|\widehat{L}_c(X) : L_c(X)|$ is (p, c, d)-bounded.*

Proof. We start by showing that the nilpotency class of $\widehat{L}_c(X)$ is c. Since $\widehat{L}_c(X)$ is a $\mathbb{Z}_p$-Lie subalgebra of $\widetilde{L}_c(X)$, the nilpotency class of $\widehat{L}_c(X)$ is at most c. Since $\widehat{L}_c(X)$ contains $L_c(X)$, the nilpotency class of $\widehat{L}_c(X)$ is at least c. This shows that the nilpotency class of $\widehat{L}_c(X)$ is precisely c.

Let us check that the $\mathbb{Z}_p$-Lie algebra $\widehat{L}_c(X)$ is powerful. By the linearity of the Lie bracket $[\cdot, \cdot]$ in $\widetilde{L}_c(X)$, it suffices to show that for every $i, j \geq 1$, the following equality holds:

$$\left[\frac{1}{p^{i-1}}\gamma_i(L_c(X)), \frac{1}{p^{j-1}}\gamma_j(L_c(X))\right] \subset p\widehat{L}_c(X).$$

Indeed, for every $i, j \geq 1$, we have that

$$\begin{aligned}\left[\frac{1}{p^{i-1}}\gamma_i(L_c(X)), \frac{1}{p^{j-1}}\gamma_j(L_c(X))\right] &= \frac{1}{p^{i+j-2}}[\gamma_i(L_c(X)), \gamma_j(L_c(X))] \\ &\subseteq p\left(\frac{1}{p^{i+j-1}}\gamma_{i+j}(L_c(X))\right) \subset p\widehat{L}_c(X).\end{aligned}$$

So, $\widehat{L}_c(X)$ is powerful.

Next, we show that the rank of $\widehat{L}_c(X)$ is bounded by a function depending on c and d. Consider $\widetilde{L}_c(X) = L_c(X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as a $\mathbb{Q}_p$-vector space and decompose it as

$$\widetilde{L}_c(X) = \mathrm{Span}(X) \oplus \mathrm{Span}([X,X]) \oplus \cdots \oplus \mathrm{Span}([X, \overset{c}{\ldots}, X]),$$

where the dimension of $\mathrm{Span}([X, \overset{i}{\ldots}, X])$ as a $\mathbb{Q}_p$-vector space is smaller than or equal to d^i. Then we have that the dimension of $\widetilde{L}_c(X)$ as a $\mathbb{Q}_p$-vector space is

$$\dim_{\mathbb{Q}_p}(\widetilde{L}_c(X)) \le d + d^2 + \cdots + d^c = \frac{d^{c+1} - d}{d-1}.$$

Notice that $\widehat{L}_c(X)$ has no torsion and recall that

$$\widehat{L}_c(X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = L_c(X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \widetilde{L}_c(X).$$

Thus, the inequality

$$\mathrm{rk}(\widehat{L}_c(X)) = \dim_{\mathbb{Q}_p}(\widetilde{L}_c(X)) \le \frac{d^{c+1} - d}{d-1}$$

proves item (b).

Finally, as $p^{c-1}\widehat{L}_c(X) \subseteq L_c(X)$, we have that $|\widehat{L}_c(X) : L_c(X)| \le p^{(c-1)r}$, where $r = \mathrm{rk}(G)$. □

Let I be an ideal of $L_c(X)$ such that $|L_c(X) : I| < \infty$. This ideal is not necessarily an ideal of $\widehat{L}_c(X)$ and thus, we extend I to an ideal $\widehat{I}$ of $\widehat{L}_c(X)$ as follows:

$$\widehat{I} := I + \frac{1}{p}[I, L_c(X)] + \frac{1}{p^2}[[I, L_c(X)], L_c(X)] + \cdots + \frac{1}{p^{c-1}}[I,_{(c-1)} L_c(X)].$$

Lemma 3.3. *Let $\widehat{L}_c(X)$ and $\widehat{I}$ be as above. Then*

$$[\widehat{I}, \widehat{L}_c(X)] \subset p\widehat{I},$$

and, in particular, $\widehat{I}$ is an ideal in $\widehat{L}_c(X)$.

Proof. For all $i \ge 1$ and $j \ge 1$, we have that

$$\begin{aligned}
\left[\frac{1}{p^i}[I,_i L_c(X)], \frac{1}{p^{j-1}}\gamma_j(L_c(X))\right] &= \frac{1}{p^{i+j-1}}[[I,_i L_c(X)], \gamma_j(L_c(X))] \\
&\subset \frac{1}{p^{i+j-1}}[[I,_i L_c(X)],_j L_c(X)] \\
&= \frac{1}{p^{i+j-1}}[I,_{i+j} L_c(X)] \\
&= p\left(\frac{1}{p^{i+j}}[I,_{i+j} L_c(X)]\right) \\
&\subset p\widehat{I},
\end{aligned}$$

where in the first inclusion we used Lemma 3.1. By the linearity property of the Lie bracket, we conclude that $[\widehat{I}, \widehat{L}_c(X)] \subset p\widehat{I}$. □

Proposition 3.4. *The quotient $\widehat{L}_c(X)/\widehat{I}$ is a finite powerful p-central $\mathbb{Z}_p$-Lie algebra with the* ΩEP *and with (c,d)-bounded rank.*

Proof. We start by showing that the quotient $\widehat{L}_c(X)/\widehat{I}$ is powerful. From part (a) of Lemma 3.2, we have that the $\mathbb{Z}_p$-Lie algebra $\widehat{L}_c(X)$ is powerful and since this property is inherited by factor Lie algebras, our claim holds. Also, the rank of $\widehat{L}_c(X)/\widehat{I}$ is at most the rank of $\widehat{L}_c(X)$ which is (c,d)-bounded by Lemma 3.2 (b).

Now, consider $p\widehat{I} \subset \widehat{I}$ which is an ideal of $\widehat{L}_c(X)$. It is straightforward to see that $\Omega_1(\widehat{L}_c(X)/p\widehat{I}) = \widehat{I}/p\widehat{I}$ and therefore,

$$\frac{\widehat{L}_c(X)/p\widehat{I}}{\Omega_1(\widehat{L}_c(X)/p\widehat{I})} = \frac{\widehat{L}_c(X)/p\widehat{I}}{\widehat{I}/p\widehat{I}} \cong \widehat{L}_c(X)/\widehat{I}.$$

Thus, to prove that $\widehat{L}_c(X)/\widehat{I}$ has the ΩEP, it is enough to show that $\widehat{L}_c(X)/p\widehat{I}$ is a p-central $\mathbb{Z}_p$-Lie algebra. Indeed, by the previous lemma $[\widehat{I}, \widehat{L}_c(X)] \subset p\widehat{I}$, which shows that

$$[\Omega_1(\widehat{L}_c(X)/p\widehat{I}), \widehat{L}_c(X)/p\widehat{I}] = [\widehat{I}/p\widehat{I}, \widehat{L}_c(X)/p\widehat{I}] = 0.$$

Hence, we conclude that $\widehat{L}_c(X)/\widehat{I}$ is a powerful p-central $\mathbb{Z}_p$-Lie algebra with the ΩEP. □

Let I be a Lie ideal of $L_c(X)$ and let $\varphi : L_c(X)/I \to \widehat{L}_c(X)/\widehat{I}$ be the natural map that sends an element $a + I$ to $a + \widehat{I}$. Then

$$\operatorname{Ker}\varphi \cong \frac{L_c(X) \cap \widehat{I}}{I} \quad \text{and} \quad \varphi\left(\frac{L_c(X)}{I}\right) \cong \frac{L_c(X)}{L_c(X) \cap \widehat{I}} \cong \frac{L_c(X) + \widehat{I}}{\widehat{I}}.$$

Thus, for any ideal I, there is an extension of $\mathbb{Z}_p$-Lie algebras

$$0 \to \frac{L_c(X) \cap \widehat{I}}{I} \hookrightarrow \frac{L_c(X)}{I} \twoheadrightarrow \frac{L_c(X) + \widehat{I}}{\widehat{I}} \to 0, \tag{3.1}$$

satisfying the properties of the following proposition.

Proposition 3.5. *Let d, c be positive integers, let X be a set with d elements and let I be an ideal of $L_c(X)$. Under the above notation, the following properties hold:*

(a) *The order $|(L_c(X) \cap \widehat{I})/I|$ is (p,c,d)-bounded.*

(b) *The index $|(\widehat{L}_c(X)/\widehat{I}) : (L_c(X) + \widehat{I}/\widehat{I})|$ is (p,c,d)-bounded.*

Proof. We start by proving that the order $|(L_c(X)\cap\widehat{I})/I|$ is (p,c,d)-bounded. To that end, we shall show that both its rank and its exponent are (p,c,d)-bounded. Let r denote the rank of $\widehat{L}_c(X)$ and note that $\mathrm{rk}((L_c(X)\cap\widehat{I})/I)\leq r$. It follows from Lemma 3.2 (b) that r is (c,d)-bounded and thus, $\mathrm{rk}((L_c(X)\cap\widehat{I})/I)$ is also (c,d)-bounded. Moreover, as $L_c(X)\cap\widehat{I}\subseteq\widehat{I}$ and $p^{c-1}\widehat{I}\subseteq I$, the exponent of $(L_c(X)\cap\widehat{I})/I$ is also (p,c)-bounded. Thus, the first claim holds.

The last claim follows from the fact that

$$\left|\frac{\widehat{L}_c(X)}{\widehat{I}}:\frac{L_c(X)+\widehat{I}}{\widehat{I}}\right| = |\widehat{L}_c(X):L_c(X)+\widehat{I}|\leq|\widehat{L}_c(X):L_c(X)|,$$

and from Lemma 3.2 (c). □

Theorem 3.6. *Let p be a prime number and let L be a d-generated finite $\mathbb{Z}_p$-Lie algebra of nilpotency class c. Then there exist a powerful p-central $\mathbb{Z}_p$-Lie algebra $\widehat{L}$ with (d,c)-bounded number of generators and an ideal J of L such that*

(a) $|J|$ *is (p,c,d)-bounded,*

(b) L/J *can be embedded as a subalgebra in $\widehat{L}/\Omega_1(\widehat{L})$,*

(c) $|\widehat{L}/\Omega_1(\widehat{L}):L/J|$ *is (p,c,d)-bounded.*

Proof. Let X denote a generating set of the d-generated finite $\mathbb{Z}_p$-Lie algebra L of nilpotency class c. Let $L_c(X)$ denote the free $\mathbb{Z}_p$-Lie algebra of nilpotency class c and let $\pi:L_c(X)\twoheadrightarrow L$ be the projection map. Set $I:=\mathrm{Ker}\,\pi$, so that $L\cong L_c(X)/I$. Then L fits into an extension of the form (3.1).

By abusing the notation, take

$$J=\frac{L_c(X)\cap\widehat{I}}{I}\subseteq\frac{L_c(X)}{I}\cong L$$

of (p,c,d)-bounded order (see Proposition 3.5 (a)). Now,

$$L/J\cong\frac{L_c(X)}{L_c(X)\cap\widehat{I}}\cong\frac{L_c(X)+\widehat{I}}{\widehat{I}}$$

can be embedded in $\widehat{L}_c(X)/\widehat{I}$, where $|\widehat{L}_c(X)/\widehat{I}:L/J|$ is (p,c,d)-bounded.

Finally, by Proposition 3.4, $\widehat{L}_c(X)/\widehat{I}$ has the ΩEP, that is,

$$\frac{\widehat{L}_c(X)}{\widehat{I}}\cong\frac{\widehat{L}}{\Omega_1(\widehat{L})}$$

for some p-central $\mathbb{Z}_p$-Lie algebra. Thus, the result holds. □

Now we can translate this result to p-groups of small nilpotency class by the Lazard correspondence.

Theorem 3.7. *Let p be a prime number. Let G be a d-generated finite p-group of nilpotency class $c < p$. Then there exist a powerful p-central p-group $\widehat{G}$ with (d, c)-bounded number of generators and a normal subgroup N of G such that*

(a) *$|N|$ is (p, c, d)-bounded,*

(b) *G/N can be embedded as a subgroup of $\widehat{G}/\Omega_1(\widehat{G})$,*

(c) *$|\widehat{G}/\Omega_1(\widehat{G}) : G/N|$ is (p, c, d)-bounded.*

Proof. The claim follows from Theorem 3.6 and Theorem 2.3. □

4 Counting arguments

In this section, we shall prove some counting argument results using spectral sequences that will be essential to prove the main result. Actually, the following results can be considered as a generalization of [3, Lemma 4.3 and Theorem 4.4].

Lemma 4.1. *Let p be an odd prime, let r, c be integral numbers and let G be a p-group with $\mathrm{rk}(G) \leq r$. Let*

$$1 \longrightarrow C_p \longrightarrow G \xrightarrow{\pi} Q \longrightarrow 1,$$

be an extension of groups. Suppose that Q has a subgroup A of nilpotency class $c < p$. Set $B = \pi^{-1}(A)^{p^2}$. Then B is a powerful p-central p-group of nilpotency class at most c with the ΩEP and $|G : B| \leq p^{2cr}|Q : A|$.

Proof. Put $C = \pi^{-1}(A)$, $D = C^p$ and let N be the image of C_p in G. Since the nilpotency class of A is c, we know that the nilpotency class of C is at most $c + 1$. Then we may write

$$\gamma_{p-1}(D) = [\gamma_{p-2}(C^p), C^p] \leq [\gamma_{p-2}(C^p), C]^p \subseteq N^p = 1,$$

where in the inequality we used the fact that C is a regular p-group and [5, Lemma 2.13]. In particular, D is a p-group of nilpotency class at most c and by Proposition 2.4, $B = D^p$ is a powerful p-central group with the ΩEP.

We also have $|G : B| = |G : C||C : D||D : B|$ and $|G : C| = |Q : A|$, where C/D and D/B have exponent p. Moreover, C has rank at most r and nilpotency class at most $c + 1$. We may write

$$|C : D| = |C : D\gamma_2(C)||D\gamma_2(C) : D\gamma_3(C)| \cdots |D\gamma_c(C) : D|.$$

Note that the quotients $D\gamma_i(C)/D\gamma_{i+1}(C)$ are elementary abelian and therefore,

$$|D\gamma_i(C) : D\gamma_{i+1}(C)| \leq p^r.$$

In particular, $|C : D| \leq p^{cr}$. A similar argument shows that $|D : B| \leq p^{cr}$. Then the bound in the statement follows. □

We continue with the main result of this section which is a generalization of [1, Theorem 3.3].

Theorem 4.2. *Let p be an odd prime, let c, r, n, f be positive integers and suppose that*

$$1 \to H \to G \to Q \to 1$$

is an extension of finite p-groups with $|H| \leq p^n$, $\mathrm{rk}(G) \leq r$ and Q has a subgroup A of nilpotency class $c < p$ with $|Q : A| \leq f$. Then the ring $H^*(G; \mathbb{F}_p)$ is determined up to a finite number of possibilities (depending on p, n, r and f) by the ring $H^*(Q; \mathbb{F}_p)$.

Proof. We start with the base case $H \cong C_p$. By Lemma 4.1, there exist a powerful p-central subgroup B of G with ΩEP and whose index is bounded in terms of p, r and f. If H is not contained in B, we consider $H \times B$ instead of B. In both situations, there exists an element $\eta \in H^2(B, \mathbb{F}_p)$ (respectively, $\eta \in H^2(B \times H, \mathbb{F}_p)$) such that $\mathrm{res}^B_H(\eta)$ is non-zero (respectively, $\mathrm{res}^{B \times H}_H(\eta)$) (see Theorem 2.2). Then the spectral sequence arising from $1 \to H \to G \to Q \to 1$ stops at most at the page $2|G : B| + 1$ (cf. [1, proof of Lemma 3.2]). Now, by [1, Proposition 3.1], the theorem holds.

For general H, we proceed by induction on $|H|$. Suppose that the result holds for all the group extensions of the form

$$1 \to H' \to G \to Q,$$

where $|H'| < |H| \leq p^n$, $\mathrm{rk}(G) \leq r$ and with $A \leq Q$ of nilpotency class $c < p$ and $|Q : A| \leq f$. Choose a subgroup $H' \leq H$ with $H' \trianglelefteq G$ and $|H : H'| = p$. The quotients $G' = G/H'$ and $C_p \cong H/H'$ fit in a short exact sequence

$$1 \longrightarrow C_p \longrightarrow G' \xrightarrow{\pi} Q \longrightarrow 1 \tag{4.1}$$

and we also have the following extension of groups:

$$1 \to H' \to G \to G' \to 1. \tag{4.2}$$

Applying Lemma 4.1 to the extension of groups (4.1), we know that G' has a powerful p-central p-subgroup B' with the ΩEP with $|G' : B'| \leq p^{2cr}|Q : A|$. Also, by the previous case, we have that the cohomology algebra $H^*(G'; \mathbb{F}_p)$ is determined up to a finite number of possibilities (depending on p, n, r and f) by the algebra $H^*(Q; \mathbb{F}_p)$.

Now, we may apply the induction hypothesis to the extension (4.2) since $|H'| < |H|$. Then the cohomology algebra $H^*(G; \mathbb{F}_p)$ is determined up to a finite number of possibilities (depending on p, n, r and f) by the algebra $H^*(G'; \mathbb{F}_p)$. In turn, the result holds. □

5 Main result and further work

In this section we prove the main result of this paper and we mention the main obstructions to extending this result.

Theorem 5.1. *Let p be a prime number and let d be a non-negative integer. Then the number of possible isomorphism types for the mod p-cohomology algebra of a d-generated p-group of nilpotency class smaller than p is bounded by a function depending only on p and d.*

Proof. For $p = 2$ the statement is clear since G is a d-generated finite abelian p-group. Assume now that p is odd. Let G be a d-generated finite p-group of nilpotency class smaller than p. Notice that the elementary abelian quotients $\gamma_i(G)G^p/\gamma_{i+1}(G)G^p$ can be generated by at most d^i commutators. Therefore,

$$|G : G^p| = |\gamma_1(G)G^p : \gamma_2(G)G^p||\gamma_2(G)G^p : \gamma_3(G)G^p| \cdots |\gamma_{p-1}(G)G^p : G^p|$$
$$\leq p^{d+d^2+\cdots+d^{p-1}}.$$

Since G is a regular p-group (see [5, Theorem 2.8 (i)] or [8, Corollary 4.13]), for any subgroup H of G, we have that

$$|H : \Phi(H)| \leq |H : H^p| = |\Omega_1(H)| \leq |\Omega_1(G)| = |G : G^p|$$
$$\leq p^{d+d^2+\cdots+d^{p-1}}.$$

In particular, $\mathrm{rk}(G) \leq d + d^2 + \cdots + d^{p-1}$ (see [5, Theorem 2.8 (i) and Theorem 2.10 (iv)] or [8, Theorems 4.21, 4.26, 4.3]).

By Theorem 3.7, there exist a powerful p-central group $\widehat{G}$ and a normal subgroup N of G such that the following hold:

(i) The number of generators of $\widehat{G}$ is (d)-bounded.

(ii) The order $|N|$ is (p, d)-bounded.

(iii) G/N can be embedded as a subgroup of $\tilde{G} = \widehat{G}/\Omega_1(\widehat{G})$ whose index is (p, d)-bounded.

Since $\tilde{G}$ is a powerful p-central p-group, the number of generators of $\tilde{G}$ coincides with the number of generators of $\Omega_1(\tilde{G})$ (compare [7, Theorem 6.5] and the facts that $\tilde{G}$ is powerful and $\Omega_1(\tilde{G})$ is abelian). Furthermore, $\tilde{G}$ has the ΩEP. Therefore, by Theorem 2.2, $H^*(\tilde{G}; \mathbb{F}_p)$ is isomorphic to the graded $\mathbb{F}_p$-algebra $\Lambda(y_1, \ldots, y_e) \otimes \mathbb{F}_p[x_1, \ldots, x_e]$, where e denotes the number of generators of $\tilde{G}$ and $|y_i| = 1$ and $|x_i| = 2$. Recall that the possibilities for e are bounded in terms of p and d (e is at most $d + d^2 + \cdots + d^{p-1}$). By Theorem [1, Theorem 3.5] and point (iii) above, the number of isomorphism types for $H^*(G/N; \mathbb{F}_p)$ is bounded in terms of p and d.

Finally, since the rank of G is (p,d)-bounded, the nilpotency class of G/N is smaller than p and $|N|$ is (p,d)-bounded, we may apply Theorem 4.2 to the extension

$$1 \to N \to G \to G/N \to 1,$$

with $A = G/N$, to conclude that the number of isomorphism types of algebras for $H^*(G;\mathbb{F}_p)$ is bounded in terms of p and d. This concludes the proof of the main result of this paper. □

The main obstruction to extending this result to p-groups of arbitrary nilpotency class is the absence of a nice Lie theory for such groups. However, we think that an analogous result to that of Theorem 3.7 should hold for p-groups of arbitrary nilpotency class c. Thus, we propose the following conjecture.

Conjecture 5.1. Let p be a prime number and let c and d be non-negative integers. Then the number of possible isomorphism types for the mod p cohomology algebra of a d-generated p-group of nilpotency class c is bounded by a function depending on p, d and c.

Finite p-groups of fixed nilpotency class or fixed coclass c share one common property, namely, that their ranks are bounded by some function depending on p and c. The following conjecture encompasses the above conjecture as well as Carlson's conjecture for the cohomology of finite p-groups of fixed coclass (see [1, Section 6]).

Conjecture 5.2. Let p be prime number and let r be a non-negative integer. Then the number of possible isomorphism types for the mod p cohomology algebra of a p-group of rank r is bounded by a function depending on p and r.

One argument in favor of this conjecture is that if G is a p-group of rank r, then it contains a powerful p-central subgroup with the ΩEP whose index is bounded in terms of p and r. The existence of a powerful p-central subgroup is classical (see, for example, [9, Section 11]). The fact that this subgroup can be chosen to have the ΩEP, at least for $p \geq 5$, can be deduced from the Ph.D. thesis of the last author [6] or from the Habilitation Schrift of T. S. Weigel [11].

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